Production, Costs, and Firm Supply

Lecture 3

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Production Functions

The Firm's Production Function

A firm turns inputs into outputs. The **production function** summarizes this:

$$Q = f(K, L)$$

- Q: output per period
- K: capital (machine-hours)
- L: labor (labor-hours)

Shows the *maximum* output from each input combination (efficient production).

Marginal Productivity

The **marginal product** of an input is the additional output from one more unit of that input, holding other inputs constant:

$$MP_L = \frac{\partial f}{\partial L} = f_L, \qquad MP_K = \frac{\partial f}{\partial K} = f_K$$

Diminishing marginal productivity: We assume that marginal products eventually decrease

$$\frac{\partial^2 f}{\partial L^2} = f_{LL} < 0, \qquad \frac{\partial^2 f}{\partial K^2} = f_{KK} < 0$$

Holding capital fixed, each additional worker adds less and less output.

Average vs. Marginal Productivity

Average product of labor:

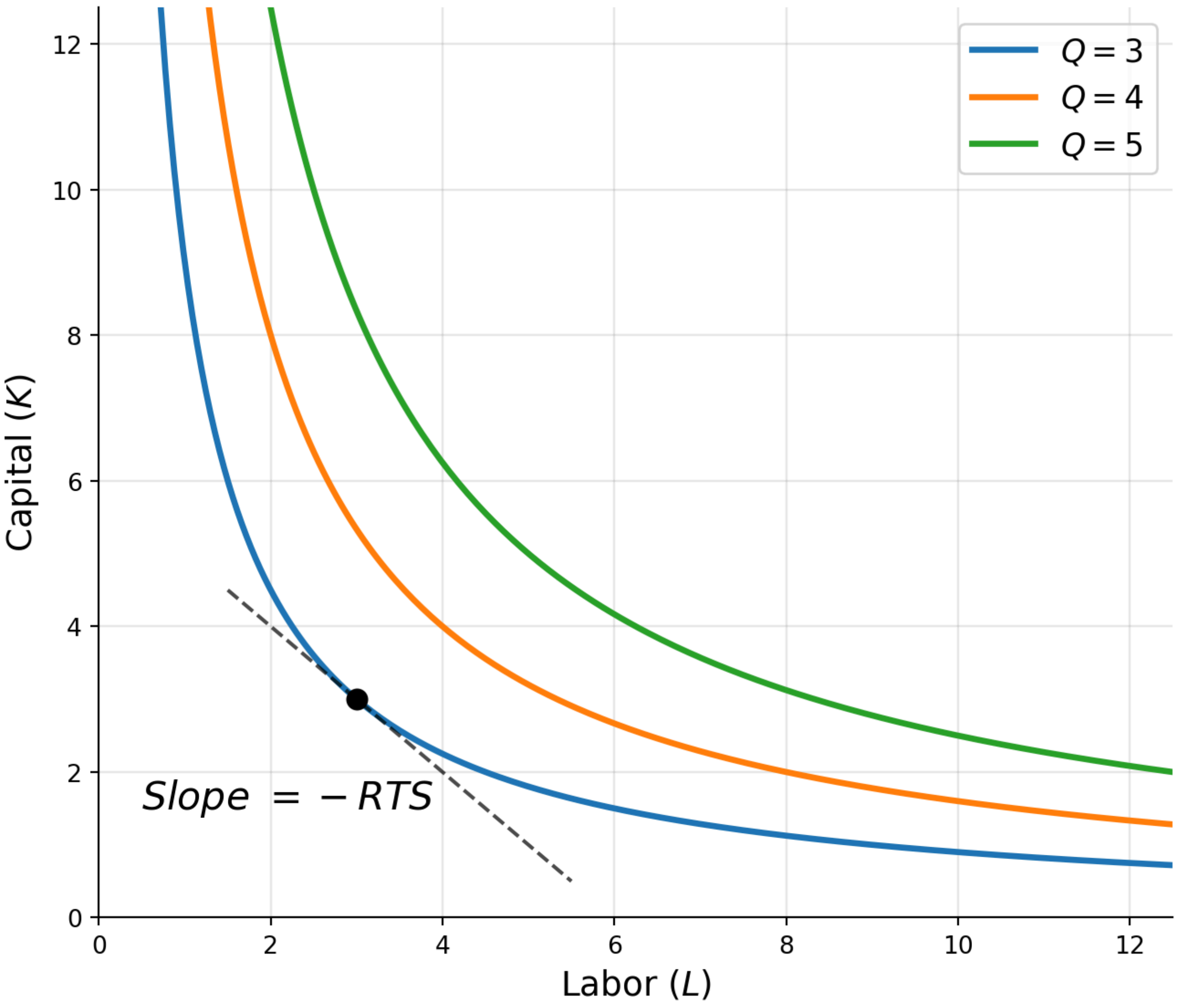
$$AP_L = \frac{Q}{L} = \frac{f(K, L)}{L}$$

- What people usually mean by “labor productivity”
- Depends on how much capital workers have

Important: AP_L depends on K too. When we say “U.S. workers are more productive than Indian workers,” much of that reflects more capital per worker, not inherently more productive labor.

Cross-productivity effects: Typically $f_{KL} > 0$ — more capital makes labor more productive and vice versa.

Isoquants and the RTS



Analogy to Consumer Theory

Consumer	Producer
Utility $U(x, y)$	Output $f(K, L)$
Budget constraint	Cost constraint
Indifference curves	Isoquants
MRS	RTS

Marginal Rate of Technical Substitution

The **RTS** (rate of technical substitution) is the rate at which labor can substitute for capital, holding output constant:

$$RTS = -\frac{dK}{dL}\bigg|_{Q=\bar{Q}} = \frac{MP_L}{MP_K} = \frac{f_L}{f_K}$$

Derivation: Total differential along an isoquant ($dQ = 0$):

$$dQ = f_L\ dL + f_K\ dK = 0 \implies \frac{dK}{dL} = -\frac{f_L}{f_K}$$

Diminishing RTS: As we move along an isoquant (more L, less K), the RTS falls. This means isoquants are **convex** — same logic as diminishing MRS for consumers.

Elasticity of Substitution

Along an isoquant, RTS decreases as capital-labor (K/L) ratio decreases

The **elasticity of substitution** σ measures responsiveness of RTS to changes in K/L:

$$\sigma = \frac{\%\Delta(K/L)}{\%\Delta RTS} = \frac{d\ln(K/L)}{d\ln(RTS)}$$

σ captures how easily the firm can substitute between inputs.

- Higher σ: inputs are easily substitutable (flatter isoquants)
- Lower σ: inputs are harder to substitute (more curved isoquants)

Key Production Functions

Function	Formula	σ
Linear	$Q = \alpha K + \beta L$	∞
Leontief	$Q = \min(\alpha K, \beta L)$	0
Cobb-Douglas	$Q = K^{\alpha} L^{\beta}$	1
CES	$Q = (\alpha K^{\rho} + \beta L^{\rho})^{\gamma/\rho}$	$\frac{1}{1-\rho}$

CES is appropriate for most applications because it nests the other three as special cases.

CES Function: Alternative Formulation

The CES function can also be written as:

$$Q = \left[\alpha K^{\frac{\sigma-1}{\sigma}} + \beta L^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

where σ is the elasticity of substitution.

Applications:

- Robots vs. human labor: σ estimates how easily firms can replace workers with automation
- Energy inputs: σ estimates substitutability between fossil fuels and renewables
- International trade: σ estimates substitutability between domestic and foreign goods in production

Returns to Scale

How does output respond when *all* inputs scale proportionally?

Condition	Returns to Scale
$f(tK, tL) = t \cdot f(K, L)$	Constant (CRS)
$f(tK, tL) < t \cdot f(K, L)$	Decreasing (DRS)
$f(tK, tL) > t \cdot f(K, L)$	Increasing (IRS)

CRS implies:

- Production function is homogeneous of degree 1
- Marginal products depend only on the *ratio* K/L, not on scale

Why this matters: Returns to scale determine the shape of long-run cost curves and industry structure.

Cost Functions

Economic vs. Accounting Costs

Economists and accountants think about costs differently:

Accounting costs: Out-of-pocket expenses, historical prices, depreciation

Economic costs: Opportunity costs i.e. the payment required to keep an input in its present use

Key differences:

- **Labor:** Both agree it is wages w per labor-hour.
- **Capital:** Accountant uses historical price + depreciation. Economist uses *rental rate* r.
- **Entrepreneurial services:** Accountant counts as profit. Economist recognizes the opportunity cost of the owner's time and funds.

Total economic cost: $$C = rK + wL$$

The Cost-Minimization Problem

$$\min_{K,L} \; rK + wL \quad \text{s.t.} \quad f(K,L) = q_0$$

Lagrangian:

$$\square = rK + wL + \lambda[q_0 - f(K,L)]$$

First-order conditions:

$$r = \lambda f_K, \qquad w = \lambda f_L, \qquad f(K,L) = q_0$$

Dividing the first two:

$$\frac{f_L}{f_K} = \frac{w}{r} \qquad \Longrightarrow \qquad RTS = \frac{w}{r}$$

Cost Minimization: Graphically

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/var/folders/tm/xj959v715k78plswzz0frfqc0000gn/T/ipykernel_74707/1879582369.py:36: UserWarning: No
artists with labels found to put in legend.  Note that artists whose label start with an
underscore are ignored when legend() is called with no argument.
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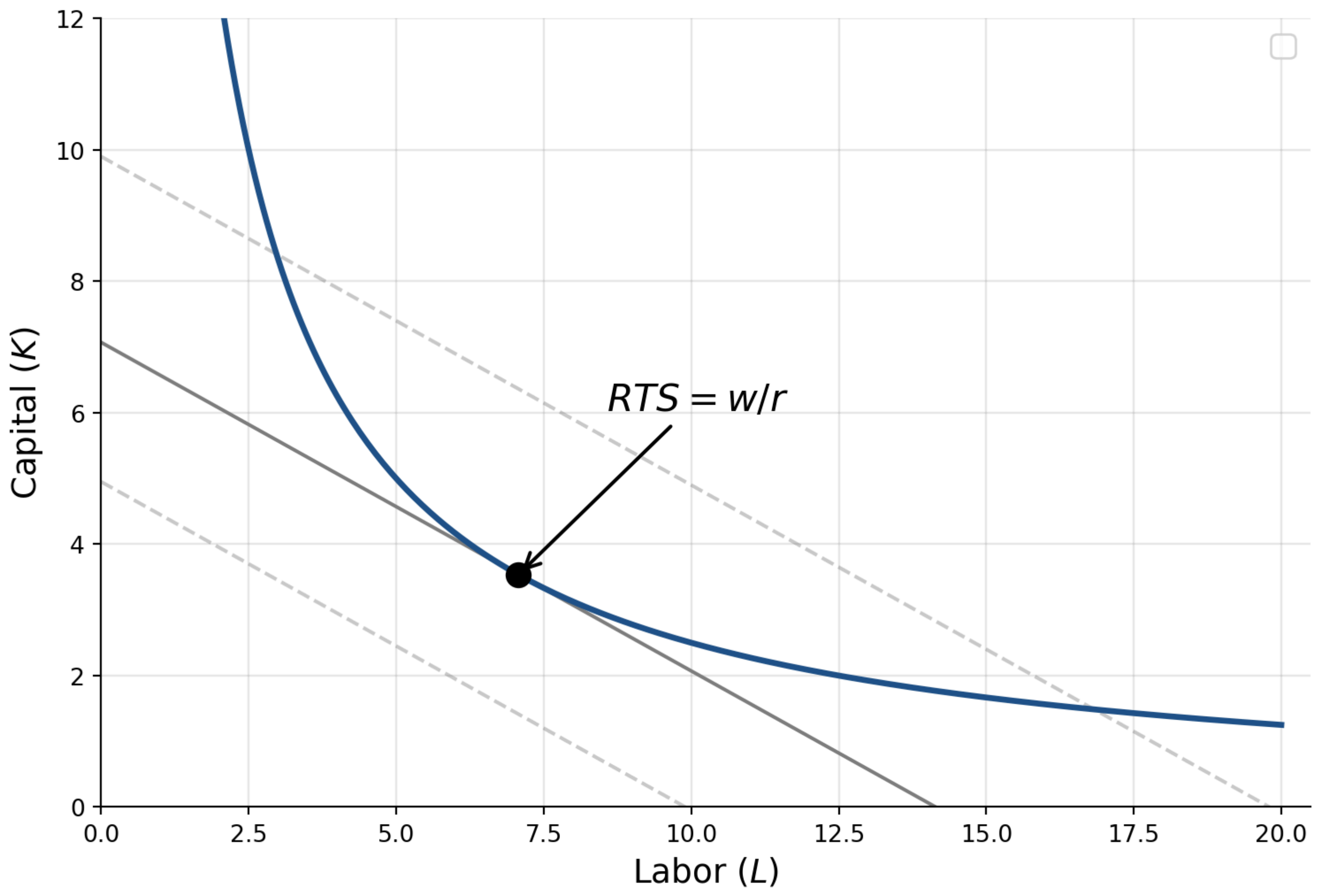


The optimal point is where the isoquant is tangent to the lowest isocost line.

Interpreting the Lagrange Multiplier

From the first-order conditions:

$$\frac{f_L}{w} = \frac{f_K}{r} = \frac{1}{\lambda}$$

Interpretation of λ:

- Marginal cost of output: $\lambda = \frac{\partial C}{\partial q_0}$
- The extra cost of producing one more unit
- Equal marginal productivity per dollar across all inputs

Cost Function

The cost function $C(r, w, Q)$ gives the minimum cost of producing Q units given input prices:

$$C(r, w, Q) = rK^*(r, w, Q) + wL^*(r, w, Q)$$

where K^* and L^* are the cost-minimizing input choices from the previous problem.

Average and Marginal Cost

Average cost: $AC = \frac{C(Q)}{Q}$

Marginal cost: $MC = \frac{\partial C}{\partial Q}$

Key relationships:

- When $MC < AC$: average cost is falling
- When $MC > AC$: average cost is rising
- When $MC = AC$: average cost is at its minimum

Intuition: Same as batting average, if your latest at-bat (marginal) is above your average, it pulls the average up.

Example: Cost Function for Cobb-Douglas

For Cobb-Douglas $Q = K^{0.5}L^{0.5}$, the cost-minimizing inputs are:

$$L^* = \frac{Q}{\sqrt{w/r}}, \qquad K^* = Q\sqrt{w/r}$$

Plugging into the cost function:

$$C(r, w, Q) = wL^* + rK^* = w \cdot \frac{Q}{\sqrt{w/r}} + r \cdot Q\sqrt{w/r} = 2Q\sqrt{rw}$$

Average and marginal cost:

$$AC = \frac{C}{Q} = 2\sqrt{rw}, \qquad MC = \frac{\partial C}{\partial Q} = 2\sqrt{rw}$$

$AC = MC$ and both are constant in Q always for CRS.

Short-Run vs. Long-Run Costs

Long run: All inputs are variable. The firm chooses the cost-minimizing (K, L).

Short run: Capital is fixed at $\bar{K}$. The firm can only adjust labor.

$$SC(\bar{K}, w, Q) = r\bar{K} + wL(Q, \bar{K})$$

- **Fixed costs ($r\bar{K}$):** incurred regardless of output
- **Variable costs (wL):** vary with output level

Key result: Short-run costs ≥ long-run costs, with equality only at the output level for which $\bar{K}$ is optimal.

$$SC(Q) \geq C(Q), \qquad SC(q) = C(q) \text{ where } \bar{K} = K^*(q)$$

Cobb-Douglas: Short-Run Costs

$$\min_{L} r\bar{K} + wL \quad \text{s.t.} \quad Q = \bar{K}^{0.5} L^{0.5}$$

In the short-run: $L^* = Q^2/\bar{K}$, so the short-run cost function is:

$$SC(\bar{K}, w, Q) = r\bar{K} + w \cdot \frac{Q^2}{\bar{K}}$$

Short-run average, marginal, and average variable cost:

$$SAC = \frac{SC}{Q} = \frac{r\bar{K}}{Q} + w \cdot \frac{Q}{\bar{K}}, \quad SMC = \frac{\partial SC}{\partial Q} = 2w \cdot \frac{Q}{\bar{K}}, \quad SAVC =$$

SAC is U-shaped because of the fixed cost component.

Profit Maximization and Supply

Profit Maximization

The firm chooses output Q to maximize **economic profit**:

$$\pi = R(Q) - C(Q) = pQ - C(Q)$$

where $R(Q)$ is total revenue and $C(Q)$ is total (economic) cost.

First-order condition:

$$\frac{d\pi}{dQ} = \frac{dR}{dQ} - \frac{dC}{dQ} = 0 \implies MR = MC$$

Second-order condition:

$$\frac{d^2\pi}{dQ^2} < 0 \implies \frac{dMR}{dQ} < \frac{dMC}{dQ}$$

At the optimum, MC must be rising faster than MR.

Marginal Revenue

Marginal revenue is the extra revenue from selling one more unit:

$$MR = \frac{dR}{dQ} = \frac{d(p(Q) \cdot Q)}{dQ} = p + Q \cdot \frac{dp}{dQ}$$

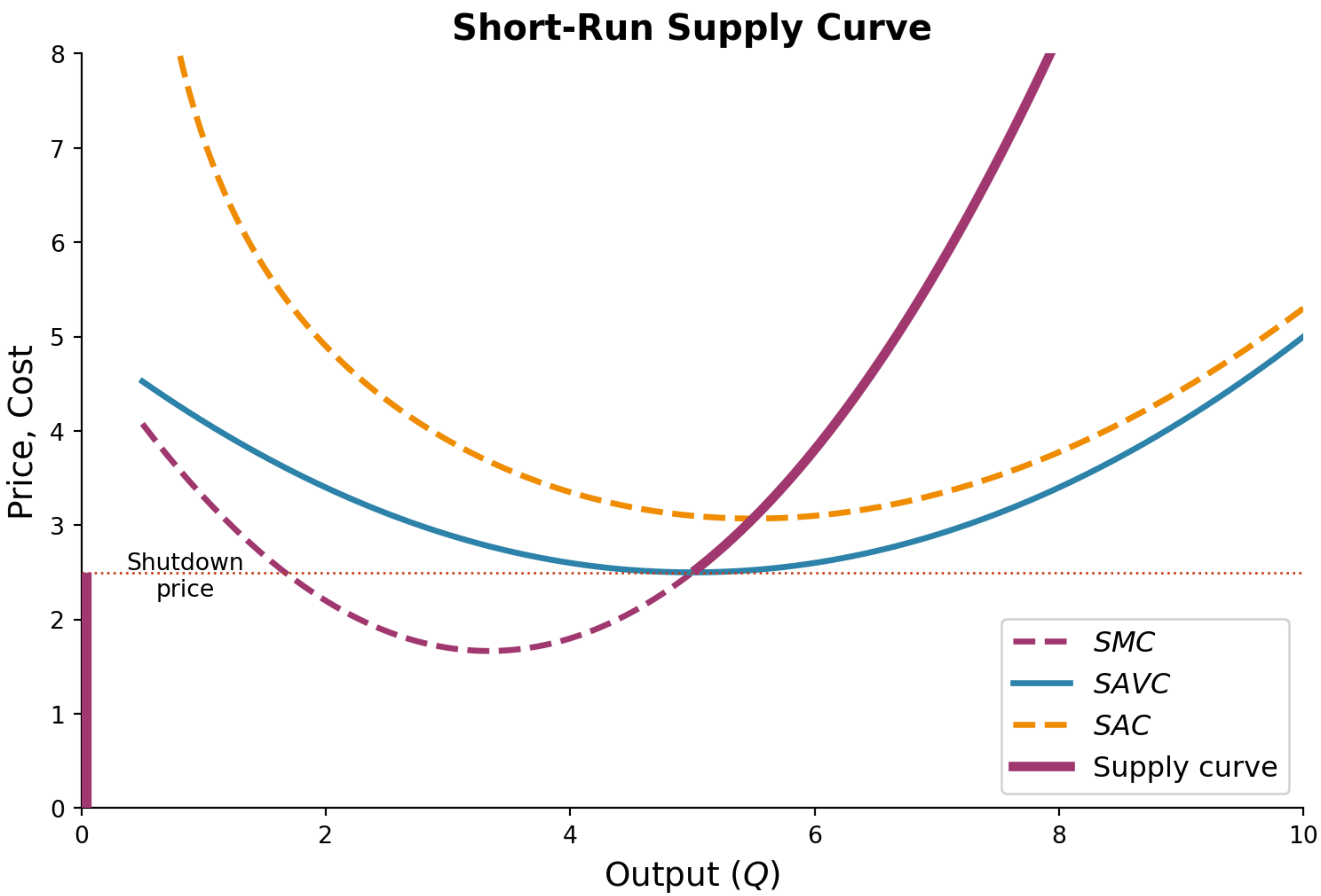
In terms of demand elasticity $e_{Q,p} = \frac{dQ}{dp} \cdot \frac{p}{Q}$, we can rewrite MR as:

$$MR = p \left(1 - \frac{1}{|e_{Q,p}|} \right)$$

For a **price-taking firm**, demand is infinitely elastic

$$|e_{Q,p}| \to \infty \implies MR \to p$$

Short-Run Supply: Price-Taking Firm



The Supply Decision

For a **price-taking firm** ($MR = P$), the profit-maximizing rule $MR = MC$ becomes

$$P = MC$$

Short-run supply curve: The upward-sloping portion of SMC above minimum $SAVC$.

The Supply Decision (cont.)

- If $P < \min(SAVC)$: produce nothing (shut down)
  - Revenue doesn't even cover variable costs
  - Losses are minimized by not producing (lose only fixed costs)
- If $\min(SAVC) \leq P < \min(SAC)$: produce, but at a loss
  - Revenue covers variable costs and some fixed costs
  - Better than shutting down
- If $P \geq \min(SAC)$: produce at a profit

Producer Surplus

Producer surplus measures the benefit to the firm from producing at the market price versus not producing at all.

$$PS = \int_{P_S}^{P} Q(P')\, dP'$$

Producer surplus is the area below the market price and above the supply curve. It’s the surplus that producers earn above their minimum willingness to supply.

Input Demand

Profit Maximization and Input Demand

We can also think of the firm as choosing inputs directly:

$$\max_{K,L} \; pf(K,L) - rK - wL$$

First-order conditions:

$$p \cdot MP_L = w \qquad \text{and} \qquad p \cdot MP_K = r$$

Hire each input until its **marginal revenue product** equals its price.

- $MRP_L = p \cdot MP_L$: extra revenue from one more unit of labor
- If $MRP_L > w$: hire more labor (adds more revenue than cost)
- If $MRP_L < w$: hire less labor

Substitution and Output Effects

What happens to labor demand when the wage w falls?

Two effects (parallel to consumer theory):

1. Substitution effect (holding output constant)

- Lower w makes labor relatively cheaper
- Firm substitutes toward labor, away from capital
- Always increases labor demand

2. Output effect (adjusting output)

- Lower w reduces marginal cost
- Firm produces more output
- More output requires more of both inputs (typically)
- Also increases labor demand

Cross-Price Effects

What about the effect of a wage decrease on *capital* demand?

$$\frac{\partial K}{\partial w} = ?$$

Substitution effect: w falls → substitute *toward* labor, *away from* capital → less K

Output effect: w falls → lower costs → more output → more K needed

These work in *opposite* directions. The net effect is ambiguous.