Demand Analysis and Consumer Welfare

Lecture 2

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Utility Maximization (contd.)

Utility Maximization

Consumer's problem:

$$\max_{x,y} U(x, y) \quad \text{subject to} \quad p_x x + p_y y = I$$

Set up the Lagrangian:

$$L = U(x, y) + \lambda(I - p_x x - p_y y)$$

First-order conditions → demand functions:

$$x^*(p_x, p_y, I) \qquad y^*(p_x, p_y, I)$$

Note: These are called **Marshallian (uncompensated) demand** functions.

Indirect Utility Function

Definition: Maximum utility as a function of prices and income

$$V(p_x, p_y, I) = \max_{x,y} U(x, y) \text{ s.t. } p_x x + p_y y = I$$

Or equivalently:

$$V(p_x, p_y, I) = U(x^*(p_x, p_y, I), y^*(p_x, p_y, I))$$

Properties:

1. Increasing in I: $\frac{\partial V}{\partial I} > 0$
2. Decreasing in prices: $\frac{\partial V}{\partial p_i} \leq 0$
3. Homogeneous of degree zero: $V(tp_x, tp_y, tI) = V(p_x, p_y, I)$

Application: Gasoline Tax

- Indirect utility as a tool for **welfare analysis** of price/income changes
- **Example**: Government raises gas tax by \$0.50/gallon. To compensate, gives everyone \$200 cash transfer.
  - To answer if people are better or worse off, compare indirect utility before and after policy.
    - Before: $V(p_{gas}, p_{other}, I)$
    - After: $V(p_{gas} + 0.50, p_{other}, I + 200)$

Question: Would it be better if we could measure welfare in dollars rather than abstract "*utils*"?

Expenditure Minimization

The Dual Problem

Two ways to think about consumer choice.

I. Utility Maximization: Given prices and income, maximize utility

$$\max_{x,y} \; U(x,y) \quad \text{s.t.} \quad p_x x + p_y y = I$$

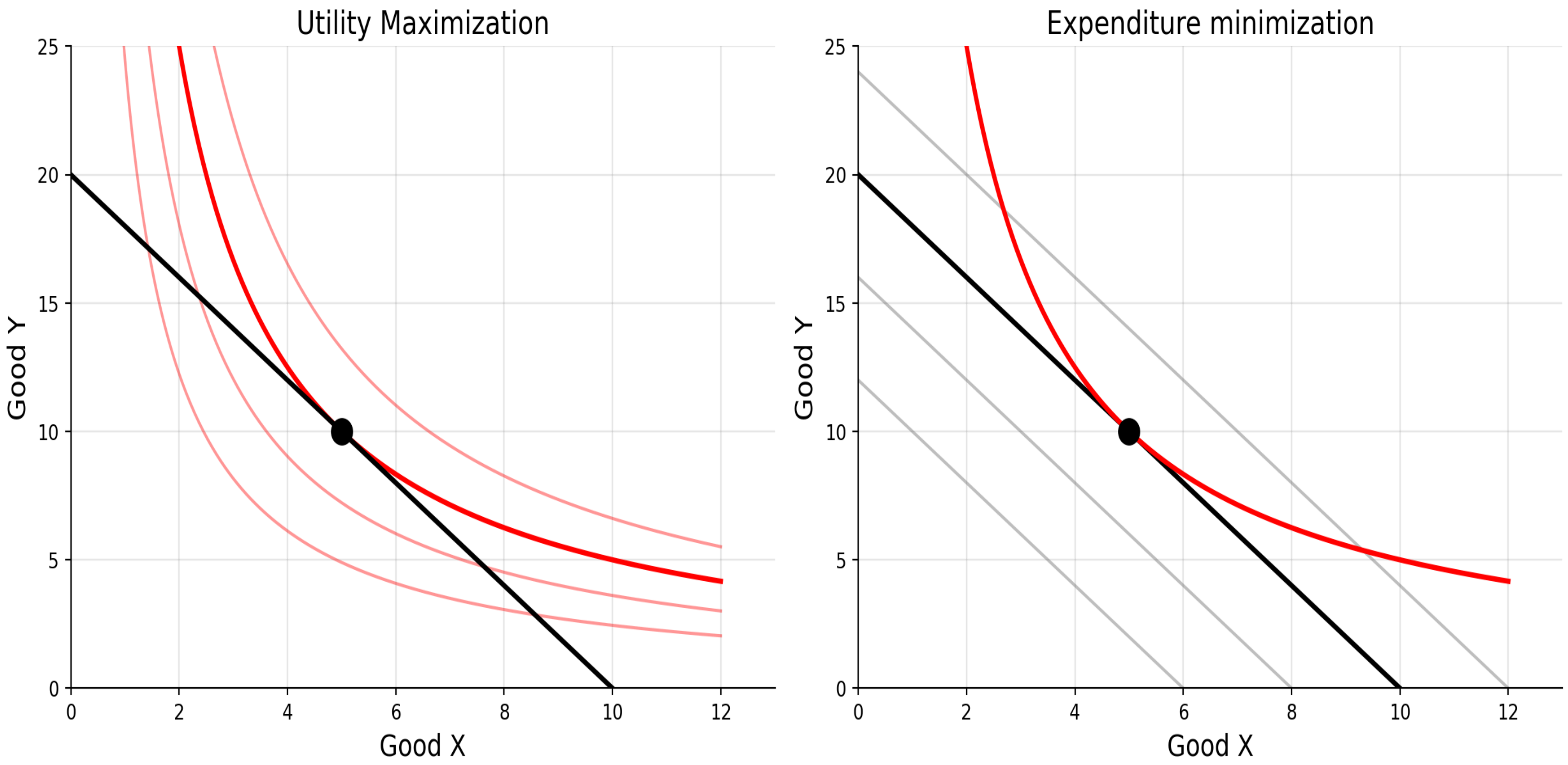
→ **Marshallian (uncompensated) demand** and **Indirect Utility Function**.

II. Expenditure Minimization: Given prices and a utility target, minimize expenditure

$$\min_{x,y} \; p_x x + p_y y \quad \text{s.t.} \quad U(x,y) = \bar{U}$$

→ **Hicksian (compensated) demand** and **Expenditure Function**.

Visualizing Duality



Expenditure Minimization

Consumer's problem:

$$\min_{x,y} p_x x + p_y y \quad \text{subject to} \quad U(x, y) = \bar{U}$$

Set up the Lagrangian:

$$L = p_x x + p_y y + \mu(\bar{U} - U(x, y))$$

FOCs give us **Hicksian (compensated) demand functions**:

$$x^h(p_x, p_y, \bar{U}) \qquad y^h(p_x, p_y, \bar{U})$$

Expenditure Function

Definition: Minimum expenditure needed to reach $\bar{U}$ at prices (p_x, p_y)

$$E(p_x, p_y, \bar{U}) = \min_{x,y} \; p_x x + p_y y \text{ s.t. } U(x, y) = \bar{U}$$

Or equivalently:

$$E(p_x, p_y, \bar{U}) = p_x x^h + p_y y^h$$

Properties:

1. Increasing in $\bar{U}$: $\frac{\partial E}{\partial \bar{U}} > 0$
2. Increasing in prices: $\frac{\partial E}{\partial p_i} \geq 0$
3. Homogeneous of degree 1 in prices: $E(tp_x, tp_y, \bar{U}) = t \cdot E(p_x, p_y, \bar{U})$
4. Concave in prices

Duality: Inverse Relationship

The indirect utility and expenditure functions are inverses.

$$V(p_x, p_y, E(p_x, p_y, \bar{U})) = \bar{U}$$

$$E(p_x, p_y, V(p_x, p_y, I)) = I$$

Practical implications:

- Can work with whichever function is more convenient
- Welfare analysis: easier to use expenditure function
- Demand analysis: easier to use indirect utility

Gasoline Tax Revisited

Example: Government raises gas tax by $0.50/gallon.

How much will the consumer need to be compensated to reach their original utility level $\bar{U}$?

$$E(p_{gas} + 0.50, p_{other}, \bar{U}) - E(p_{gas}, p_{other}, \bar{U})$$

Now we have a way of measuring how much worse off the consumer is in **dollar terms**.

Demand Functions

Two Types of Demand

Marshallian (uncompensated) demand: $x(p_x, p_y, I)$

- From utility maximization
- Holds income fixed as prices change
- Reflects both substitution and income effects

Hicksian (compensated) demand: $x^h(p_x, p_y, \bar{U})$

- From expenditure minimization
- *Compensates* the consumer for price changes to keep utility constant
- Reflects only substitution effects
- Sometimes written as x^c for "compensated"

Relationship Between Two Demands

At optimal consumption, the two demand functions give the same quantity:

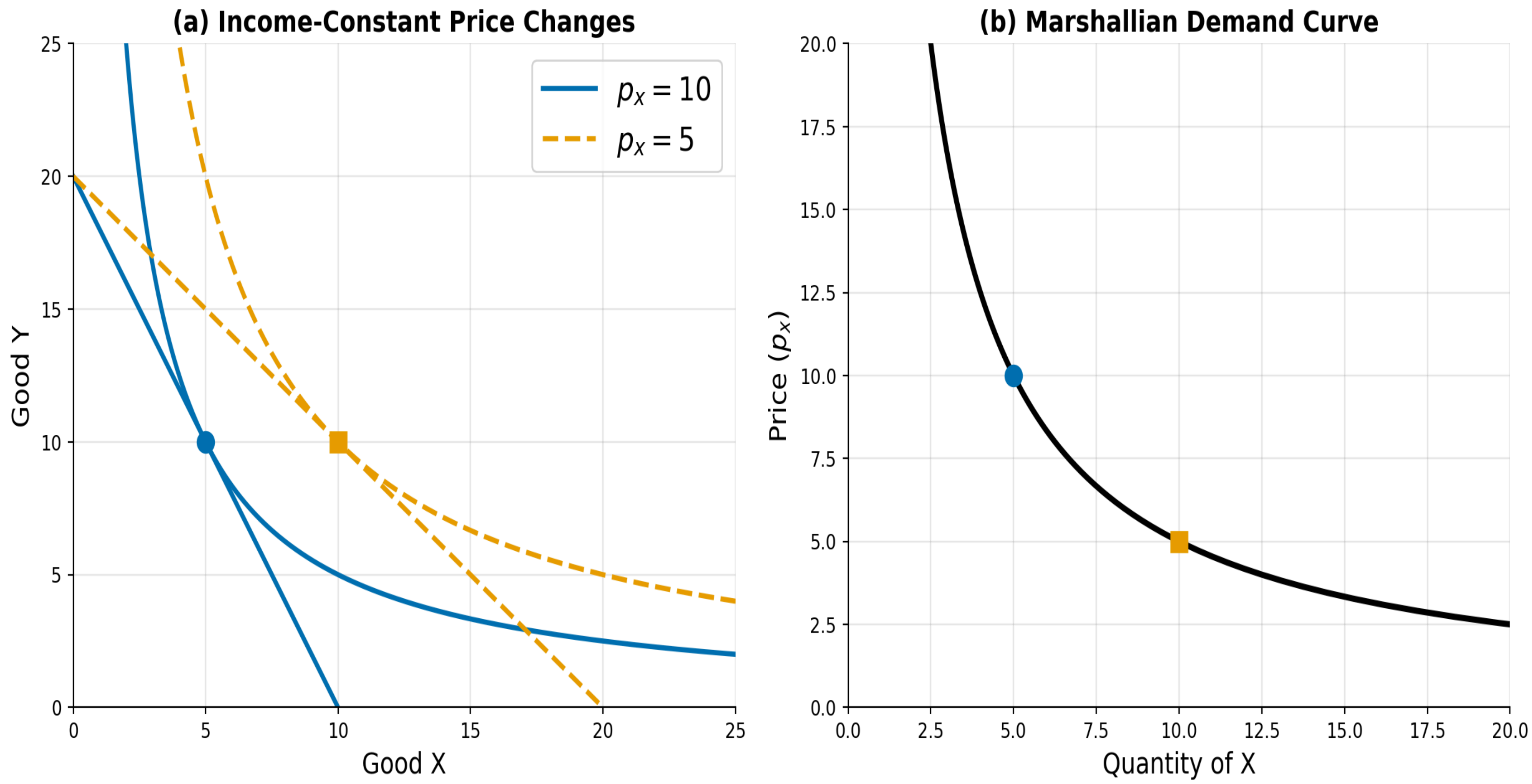
$$x(p_x, p_y, I) = x^h(p_x, p_y, V(p_x, p_y, I))$$

where $V(p_x, p_y, I)$ is the maximum utility achieved at income I.

In other words:

- Start with (p_x, p_y, I)
- Marshallian demand: $x^* = x(p_x, p_y, I)$
- Maximum utility: $\bar{U} = V(p_x, p_y, I)$
- Hicksian demand at this utility: $x^h(p_x, p_y, \bar{U}) = x^*$

Graphical: Marshallian Demand



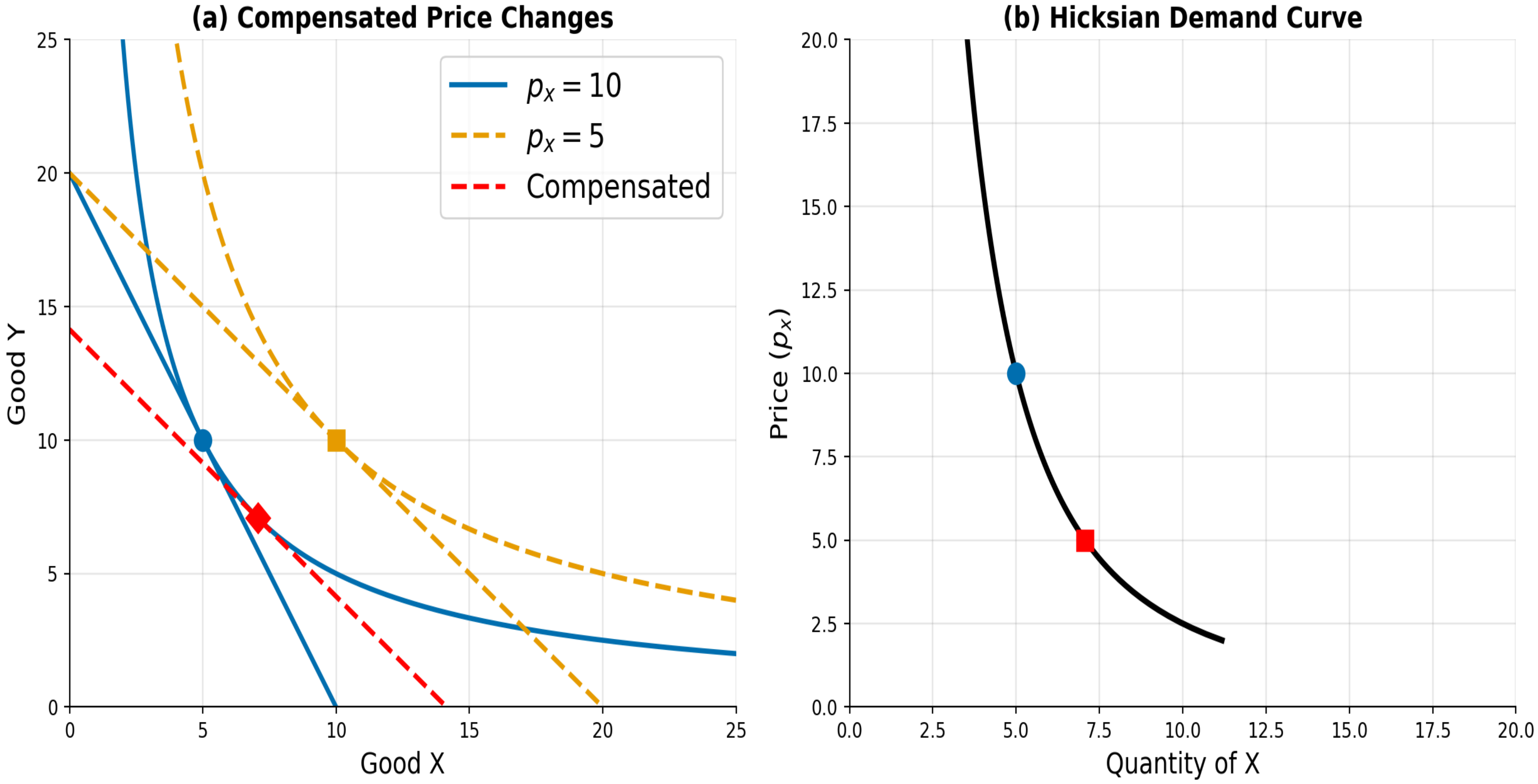
Decomposing Price Effects

When p_x changes, quantity demanded changes for two reasons:

1. **Substitution effect:** Relative prices change → reallocate consumption
2. **Income effect:** Real purchasing power changes → change consumption

Hicksian demand isolates the substitution effect by holding utility constant.

Graphical: Hicksian Demand



The Slutsky Equation

$$\underbrace{\frac{\partial x}{\partial p_x}}_{\text{Total effect}} = \underbrace{\frac{\partial x^h}{\partial p_x}}_{\text{Substitution effect (SE)}} - \underbrace{x \cdot \frac{\partial x}{\partial I}}_{\text{Income effect (IE)}}$$

Substitution effect:

- Always negative (or zero)
- Pure effect of relative price change
- Movement along indifference curve

Income effect:

- Sign depends on whether good is normal or inferior
- Effect of change in real purchasing power
- Movement to different indifference curve

Three Cases

$$\underbrace{\frac{\partial x}{\partial p_x}}_{TE} = \underbrace{\frac{\partial x^h}{\partial p_x}}_{SE \leq 0} - \underbrace{x \cdot \frac{\partial x}{\partial I}}_{IE}$$

1. Normal Good

- $\partial x / \partial I > 0$ (buy more as income rises)
- TE: Strongly negative

2. Inferior Good

- $\partial x / \partial I < 0$ (buy less as income rises)
- $|IE| < |SE| \rightarrow$ TE: Negative (but smaller than normal good)

Three Cases (Contd.)

$$\underbrace{\frac{\partial x}{\partial p_x}}_{TE} = \underbrace{\frac{\partial x^h}{\partial p_x}}_{SE \leq 0} - \underbrace{x \cdot \frac{\partial x}{\partial I}}_{IE}$$

3. Giffen Good

- $\partial x / \partial I < 0$ (inferior good)
- |IE| > |SE| → TE: *Positive* (buy less when price falls)

Demand Elasticities

Own-price elasticity (Marshallian):

$$\varepsilon_{x,p_x} = \frac{\partial x}{\partial p_x} \cdot \frac{p_x}{x}$$

Income elasticity:

$$\varepsilon_{x,I} = \frac{\partial x}{\partial I} \cdot \frac{I}{x}$$

Cross-price elasticity:

$$\varepsilon_{x,p_y} = \frac{\partial x}{\partial p_y} \cdot \frac{p_y}{x}$$

Compensated Elasticities

Compensated own-price elasticity:

$$\varepsilon^c_{x,p_x} = \frac{\partial x^h}{\partial p_x} \cdot \frac{p_x}{x^h}$$

Compensated cross-price elasticity:

$$\varepsilon^c_{x,p_y} = \frac{\partial x^h}{\partial p_y} \cdot \frac{p_y}{x^h}$$

- $\varepsilon^c_{x,p_x} \leq 0$ always (substitution effect)
- ε^c_{x,p_x} is less negative than ε_{x,p_x} for normal goods
- Compensated elasticities are symmetric: $\varepsilon^c_{x,p_y} \cdot s_x = \varepsilon^c_{y,p_x} \cdot s_y$

The Slutsky Equation in Elasticities

Dividing by x and multiplying by p_x:

$$\boxed{\varepsilon_{x,p_x} = \varepsilon^c_{x,p_x} - s_x \cdot \varepsilon_{x,I}}$$

where:

- ε_{x,p_x} = Marshallian price elasticity
- ε^c_{x,p_x} = compensated price elasticity
- $s_x = \frac{p_x x}{I}$ = budget share of good x
- $\varepsilon_{x,I}$ = income elasticity

Interpretation: Total elasticity = substitution elasticity - budget share × income elasticity

Substitutes and Complements

Gross (Marshallian) Substitutes and Complements

- Two goods x and y are **gross substitutes** if an increase in the price of y leads to an increase in the demand for x:

$$\frac{\partial x}{\partial p_y} > 0$$

- They are **gross complements** if an increase in the price of y leads to a decrease in the demand for x:

$$\frac{\partial x}{\partial p_y} < 0$$

Net (Hicksian) Substitutes and Complements

- Two goods x and y are **net substitutes** if an increase in the price of y leads to an increase in the compensated demand for x:

$$\frac{\partial x^h}{\partial p_y} > 0$$

- They are **net complements** if an increase in the price of y leads to a decrease in the compensated demand for x:

$$\frac{\partial x^h}{\partial p_y} < 0$$

Welfare

Measuring Welfare Changes

Question: How much is a consumer hurt by a price increase?

Three approaches:

1. **Compensating Variation (CV):** Money needed to restore original utility after price change
2. **Equivalent Variation (EV):** Money equivalent to price change (willingness to pay to avoid it)
3. **Consumer Surplus (CS):** Area under demand curve

All three approximate welfare changes, but differ in treatment of income effects.

Compensating Variation

Definition: Income needed to compensate for price increase to maintain original utility

For price increase from p_0 to p_1:

$$CV = E(p_1, p_y, U_0) - E(p_0, p_y, U_0)$$

where U_0 is utility before the price change.

Graphically: Area under Hicksian demand curve

$$CV = \int_{p_0}^{p_1} x^h(p, p_y, U_0)\, dp$$

Equivalent Variation

Definition: Income change equivalent to price change in terms of utility

For price increase from p_0 to p_1:

$$EV = E(p_1, p_y, U_1) - E(p_0, p_y, U_1)$$

where U_1 is utility after the price change.

***Graphically:** Area under Hicksian demand curve

$$EV = \int_{p_0}^{p_1} x^h(p, p_y, U_1)\, dp$$

(Change in) Consumer Surplus

Definition: Area under Marshallian demand curve between two prices

$$\Delta CS = \int_{p_0}^{p_1} x(p, p_y, I)\, dp$$

- **Advantage:** Easy to estimate from market data
- **Disadvantage:** Only an approximation of welfare change