Cobb-Douglas Example

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Utility Maximization

1.1 Problem Statement

The consumer's problem is:

$$\max_{x,y} \quad x^\alpha y^\beta$$

subject to:

$$p_x x + p_y y = I$$
$$x \geq 0, \quad y \geq 0$$

1.2 Lagrangian Method

Set up the Lagrangian:

$$\mathcal{L}(x, y, \lambda) = x^\alpha y^\beta + \lambda(I - p_x x - p_y y)$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = \alpha x^{\alpha-1} y^\beta - \lambda p_x = 0 \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial y} = \beta x^\alpha y^{\beta-1} - \lambda p_y = 0 \tag{2}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_x x - p_y y = 0 \tag{3}$$

1.3 Solving for Demand Functions

From equations (1) and (2):

$$\frac{\alpha x^{\alpha-1} y^\beta}{\beta x^\alpha y^{\beta-1}} = \frac{p_x}{p_y}$$

Simplifying:

$$\frac{\alpha y}{\beta x} = \frac{p_x}{p_y}$$

Solving for y:

$$y = \frac{\beta p_x}{\alpha p_y} x \tag{4}$$

Substitute equation (4) into the budget constraint (3):

$$p_x x + p_y \cdot \frac{\beta p_x}{\alpha p_y} x = I$$

$$p_x x \left(1 + \frac{\beta}{\alpha}\right) = I$$

$$p_x x \cdot \frac{\alpha + \beta}{\alpha} = I$$

Solving for x:

$$\boxed{x^*(p_x, p_y, I) = \frac{\alpha I}{(\alpha + \beta) p_x}}$$

Substituting back into equation (4):

$$y^* = \frac{\beta p_x}{\alpha p_y} \cdot \frac{\alpha I}{(\alpha + \beta) p_x} = \frac{\beta I}{(\alpha + \beta) p_y}$$

$$\boxed{y^*(p_x, p_y, I) = \frac{\beta I}{(\alpha + \beta) p_y}}$$

1.4 Indirect Utility Function

Substitute the demand functions into the utility function:

$$V(p_x, p_y, I) = \left(\frac{\alpha I}{(\alpha + \beta) p_x}\right)^{\alpha} \left(\frac{\beta I}{(\alpha + \beta) p_y}\right)^{\beta}$$

Simplifying:

$$V(p_x, p_y, I) = \frac{\alpha^\alpha \beta^\beta}{(\alpha + \beta)^{\alpha+\beta}} \cdot \frac{I^{\alpha+\beta}}{p_x^\alpha p_y^\beta}$$

Let $K = \frac{\alpha^\alpha \beta^\beta}{(\alpha+\beta)^{\alpha+\beta}}$:

$$\boxed{V(p_x, p_y, I) = K \cdot \frac{I^{\alpha+\beta}}{p_x^\alpha p_y^\beta}}$$

Expenditure Minimization

2.1 Problem Statement

The dual problem is:

$$\min_{x,y} \quad p_x x + p_y y$$

subject to:

$$x^\alpha y^\beta = \bar{U}$$

$$x \geq 0, \quad y \geq 0$$

where $\bar{U}$ is a target utility level.

2.2 Lagrangian Method

Set up the Lagrangian:

$$\mathcal{L}(x, y, \mu) = p_x x + p_y y + \mu(\bar{U} - x^\alpha y^\beta)$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = p_x - \mu \alpha x^{\alpha-1} y^\beta = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial y} = p_y - \mu \beta x^\alpha y^{\beta-1} = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = \bar{U} - x^\alpha y^\beta = 0 \tag{7}$$

2.3 Solving for Hicksian Demand

From equations (5) and (6):

$$\frac{p_x}{p_y} = \frac{\mu \alpha x^{\alpha-1} y^\beta}{\mu \beta x^\alpha y^{\beta-1}} = \frac{\alpha y}{\beta x}$$

Solving for y:

$$y = \frac{\beta p_x}{\alpha p_y} x \tag{8}$$

Substitute equation (8) into the utility constraint (7):

$$x^\alpha \left(\frac{\beta p_x}{\alpha p_y} x \right)^\beta = \bar{U}$$

$$x^{\alpha+\beta} \left(\frac{\beta p_x}{\alpha p_y} \right)^\beta = \bar{U}$$

$$x^{\alpha+\beta} = \bar{U} \left(\frac{\alpha p_y}{\beta p_x} \right)^\beta$$

$$x = \bar{U}^{1/(\alpha+\beta)} \left(\frac{\alpha p_y}{\beta p_x}\right)^{\beta/(\alpha+\beta)}$$

Rearranging:

$$\boxed{x^h(p_x, p_y, \bar{U}) = \left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} \left(\frac{p_y}{p_x}\right)^{\beta/(\alpha+\beta)} \bar{U}^{1/(\alpha+\beta)}}$$

Similarly, substituting back:

$$\boxed{y^h(p_x, p_y, \bar{U}) = \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)} \left(\frac{p_x}{p_y}\right)^{\alpha/(\alpha+\beta)} \bar{U}^{1/(\alpha+\beta)}}$$

2.4 Expenditure Function

The expenditure function is:

$$E(p_x, p_y, \bar{U}) = p_x x^h + p_y y^h$$

Substituting the Hicksian demands:

$$E = p_x \left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} \left(\frac{p_y}{p_x}\right)^{\beta/(\alpha+\beta)} \bar{U}^{1/(\alpha+\beta)} + p_y \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)} \left(\frac{p_x}{p_y}\right)^{\alpha/(\alpha+\beta)} \bar{U}^{1/(\alpha+\beta)}$$

Simplifying the first term:

$$p_x \left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} \left(\frac{p_y}{p_x}\right)^{\beta/(\alpha+\beta)} = \left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} p_x^{\alpha/(\alpha+\beta)} p_y^{\beta/(\alpha+\beta)}$$

Similarly for the second term:

$$p_y \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)} \left(\frac{p_x}{p_y}\right)^{\alpha/(\alpha+\beta)} = \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)} p_x^{\alpha/(\alpha+\beta)} p_y^{\beta/(\alpha+\beta)}$$

Therefore:

$$E = p_x^{\alpha/(\alpha+\beta)} p_y^{\beta/(\alpha+\beta)} \bar{U}^{1/(\alpha+\beta)} \left[\left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)}\right]$$

Let:

$$C = \left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)}$$

Then:

$$\boxed{E(p_x, p_y, \bar{U}) = C \cdot p_x^{\alpha/(\alpha+\beta)} p_y^{\beta/(\alpha+\beta)} \bar{U}^{1/(\alpha+\beta)}}$$