Solving Systems of Linear Equations

ECON 441: Introduction to Mathematical Economics
Lecture 7

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A Simple Economic Model

Two equations in two unknowns:

$$
\begin{aligned}
q + 2p &= 100 \\
q - 3p &= 20
\end{aligned}
$$

Can write this as:

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

Solution using Matrix Inversion

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

Cramer's Rule

More efficient way of solving a system of equations

The kth element of x can be solved by:

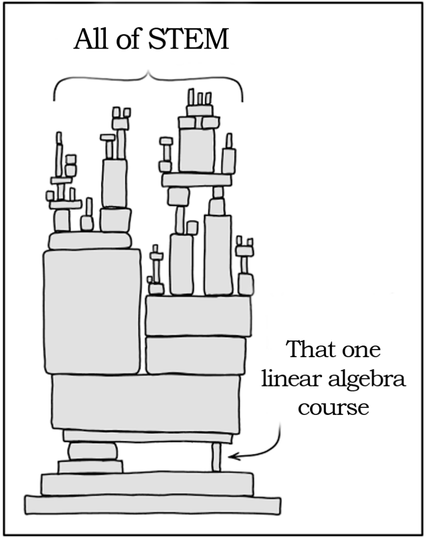
$$x_k^* = \frac{|A_k|}{|A|}$$

where A_k is a matrix formed by exchanging kth column of A by b.

Solution using Cramer's Rule

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

Coming up: Applications of Matrix Algebra



Network Theory

A network of connections can be expressed as an adjacency matrix.

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}$$

where

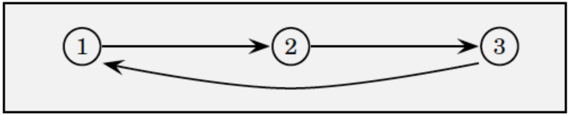
$$m_{ij} = \begin{cases} 1 & \text{if there is a direct link from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

Network Theory

Consider the following network:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrices M, M^2, M^3 give the nodes reachable in one, two and threesteps from any initial node.



Network Theory

Consider the sum:

$$S_k = M + M^2 + M^3 + \dots + M^k$$

The (i, j) element of S_k gives the number of paths of length k orless, from i to j.

For the previous example:

$$S_3 = M + M^2 + M^3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

So there is one way to go from any node to any other in three or fewer steps.

Uses of Network Theory

Network theory can be used to model
Interconnectedness of financial institutions (to predict risk of banking collapses)

Interconnectedness of the countries in world trade

Predicting supply chain risk

Markov Chain

A Markov Chain can model the transition between different states.

Example: Employment (E) and Unemployment (U).

Transition matrix:

$$P = \begin{pmatrix} P(E \to E) & P(U \to E) \\ P(E \to U) & P(U \to U) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$$

Say the initial state vector is:

$$\pi(0) = \begin{bmatrix} \pi_E(0) \\ \pi_U(0) \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

Transition Matrix

After one period, the state distribution is:

$$\pi(1) = \begin{bmatrix} \pi_E(1) \\ \pi_U(1) \end{bmatrix} = P\pi(0) = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.76 \\ 0.24 \end{bmatrix}$$

After t periods:

$$\pi(t) = P^t \pi(0)$$

Ordinary Least Squares

Linear model with k variables:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_k X_{ik} + \varepsilon_i$$

where $i = 1, \ldots, n$ denotes n observations.
Denote

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{n\times 1}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}_{k\times 1}, X = \begin{bmatrix} 1 & X_{11} & \cdots & X_{1k} \\ 1 & X_{12} & \cdots & X_{2k} \\ \vdots & \vdots & & \\ 1 & X_{1n} & \cdots & X_{nk} \end{bmatrix}_{n\times k}, \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n\times 1}$$

Then,

$$Y = X\beta + \varepsilon \qquad \text{OLS estimator: } \hat{\beta} = (X^T X)^{-1} X^T Y$$

Natural Language Processing (NLP)

Bag of Words (BoW) model is a simple and widely used method in NLP

Transform text into fixed-length vectors by counting how many times each word appears in a document
Example:

- Doc1: “the cat sat on the mat”
- Doc2: “the dog sat on the log”
- Vocabulary for these documents: $[the, cat, sat, on, mat, dog, log]$
- Vector for Doc1: $[2, 1, 1, 1, 1, 0, 0]$
- Vector for Doc2: $[2, 0, 1, 1, 0, 1, 1]$

What's next?

Midterm 1 is next week: review on Monday, exam on Wednesday

A sample exam and a help sheet are posted on the Exams page

Notes for reviewing Linear Algebra are on the website

After the midterm, we move on to differential calculus

Homework Questions

- Exercise 5.4: 6
- Exercise 5.5: 1, 2, 3 (a), (d)

Textbook reference: 5.4, 5.5, 4.7