The Determinant and the Inverse

ECON 441: Introduction to Mathematical Economics
Lecture 6

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Determinant

Determinant $|A|$ is a unique scalar associated with a **square** matrix A.

Determinant of a 2×2 Matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Can be calculated as:

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of a 3×3 Matrix

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant of a $n \times n$ Matrix

A **minor** of the element a_{ij}, denoted by $|M_{ij}|$ is obtained by deleting the ith row and jth column.

Cofactor C_{ij} is defined as:

$$|C_{ij}| = (-1)^{i+j}|M_{ij}|$$

Then,

$$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}| = \sum_{j=1}^{n} a_{ij}|C_{ij}|$$

Find the Determinant

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

Properties of Determinants

- $|A| = |A^T|$
- Interchanging rows or columns will alter the sign but not the value
- Multiplication of any one row (or one column) by a scalar k will change the value of the determinant k-fold
- The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) will leave the determinant unaltered
- If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero.

Criteria for Nonsingularity

The following statements are equivalent:

- $|A| \neq 0$
- Rows (or equivalently columns) of A are independent
- A is nonsingular
- A^{-1} exists
- A unique solution to $Ax = b$ ($x^* = A^{-1}b$) exists

Matrix Inversion

Adjoint of a nonsingular $n \times n$ matrix:

$$adjA = C^T = \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \vdots & \vdots & \cdots & \vdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix}$$

The **inverse** of A is defined as:

$$A^{-1} = \frac{1}{|A|} adjA$$

Find the Inverse

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

Find the Inverse

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

Homework Questions

- Exercise 5.2: 1 (c), (e), (f), 2, 3, 6
- Exercise 5.3: 1, 4, 5, 8
- Exercise 5.4: 2, 3, 4, 7

Textbook reference: 5.2-5.4