Matrix Multiplication, Vectors, and Special Matrices

ECON 441: Introduction to Mathematical Economics
Lecture 4

Div Bhagia

By the way


HEY A.I. WHY DO YOU ALWAYS WEAR THAT MASK?
A.I.
A.I.
A.I.
$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$
LET'S KEEP THIS ON.
A.I.
OWLTURD.COM

Matrix Multiplication

Only possible to multiply two matrices, $A_{m\times n}$ and $B_{p\times q}$ to get AB if $n = p$ i.e.

$$\text{number of columns in } A = \text{ number of rows in } B$$

So how to actually multiply these matrices?

$$C = AB$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

The **element** c_{ij} is obtained by multiplying term-by-term the entries of the i**th row of** A and j**th column of** B.

Matrix Multiplication: Examples

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$$

Here,

$$C = AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}_{2\times 2}$$

Matrix Multiplication: Examples

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2\times 3} \qquad B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}_{2\times 2}$$

AB is not defined here (why?), but BA is. Find BA.

Matrix Multiplication: Examples

One more.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1} \qquad B = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}_{1\times 3}$$

Both AB and BA are defined. Find both. What are their dimensions?

A Simple Economic Model

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

What is Ax?

$$Ax = \begin{bmatrix} q + 2p \\ q - 3p \end{bmatrix}$$

Setting $Ax = b$ gives us back our demand and supply equations.

Vectors

Matrices with only one column: column vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrices with only one row: row vectors

$$x^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

Linear Dependence

A set of vectors is said to be **linearly dependent** if and only if any one of them can be expressed as a linear combination of the remaining vectors.

Example.

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Linear Dependence

A set of vectors is said to be **linearly dependent** if and only if any one of them can be expressed as a linear combination of the remaining vectors.

Example.

$$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Linear Dependence

A set of m-vectors $v_1, v_2, \ldots, v_n$ is linearly dependent if and only if there exists a set of scaler $k_1, k_2, \ldots, k_n$ (not all zero) such that:

$$\sum_{i=1}^{n} k_i v_i = 0 \quad (m \times 1)$$

Identity Matrices

Square matrix with 1s in its **principal diagonal** and 0s elsewhere

A 2×2 identity matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A 3×3 identity matrix:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity Matrices

Acts like 1,

$$AI = IA = A$$

Example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & 2 \end{bmatrix}$$

Idempotent Matrices

A matrix is an **idempotent** matrix if it remains unchanged when multiplied by itself any number of times.

A is idempotent if and only if $A = A^k$.

Is an identity matrix idempotent?

Null Matrix

A null matrix is a matrix with all elements 0.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A + 0 = A$

$A0 = 0$

Transpose of a Matrix

Transpose of A (A^T, also written A'): interchange rows and columns

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & 2 \end{bmatrix}$$

Transpose of a Matrix

A matrix A is said to be **symmetric** if

$$A^T = A$$

A matrix A is said to be **skew-symmetric** if

$$A^T = -A$$

A matrix A is said to be **orthogonal** if

$$A^T A = I$$

Example: Symmetric Matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -5 \\ 0 & -5 & 4 \end{bmatrix}$$

Example: Skew-symmetric Matrix

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

Example: Orthogonal Matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Properties of Transposes

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

Example: $A = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$

Homework Questions

- Exercise 4.4: 5, 7
- Exercise 4.5: 1, 4
- Exercise 4.6: 2, 6
- Exercise 5.1: 3, 4

Textbook reference: 4.2-4.6