Matrices and Matrix Operations

ECON 441: Introduction to Mathematical Economics
Lecture 3

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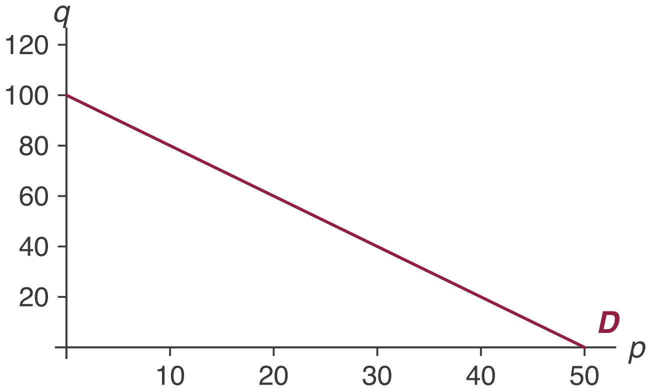
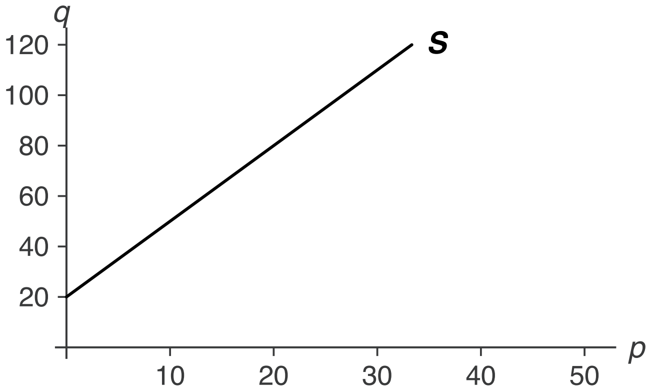
A Simple Economic Model

q: quantity of hats, p: price of a single hat
Demand for hats: $q = 100 - 2p$



A Simple Economic Model

q: quantity of hats, p: price of a single hat
Supply for hats: $q = 20 + 3p$



A Simple Economic Model

q: quantity of hats, p: price of a single hat

Demand for hats:

$$q = 100 - 2p$$

Supply for hats:

$$q = 20 + 3p$$

Equilibrium: At what price will both demand and supply be equal?

Equilibrium

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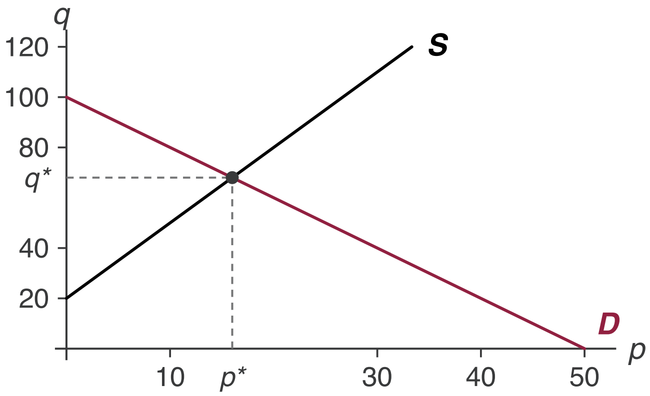
$$100 - 2p = 20 + 3p \longrightarrow p^* = \$16$$

What is the quantity traded at this price?

$$q^* = 100 - 2 \times 16 = 20 + 3 \times 16 = 68$$

q^* and p^* are determined simultaneously.

Equilibrium



Matrix Algebra

We solved a **system** of two (linear) equations in two variables.

Complex economic models: multiple equations with multiple variables

Hard to just wing it... Enter, Matrix Algebra!

Matrix Algebra can help us write complex system of equations compactly and solve them

A Simple Economic Model

q: quantity of hats, p: price of a single hat

Demand for hats: $q = 100 - 2p$

Supply for hats: $q = 20 + 3p$

Rewrite the two equations:

$$\begin{aligned} q + 2p &= 100 \\ q - 3p &= 20 \end{aligned}$$

A Simple Economic Model

Two equations in two unknowns:

$$
\begin{aligned}
q + 2p &= 100 \\
q - 3p &= 20
\end{aligned}
$$

Can write this as $Ax = b$, where

$$
A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}
$$

These arrays are called **matrices**.

Matrices

A **matrix** is a rectangular array of numbers, parameters, or vectors.

Example. $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 6 \end{bmatrix}$

Dimensions of matrix:

- Number of rows (m)
- Number of columns (n)

Dimension of a Matrix

A matrix with m rows and n columns is referred to as an $m \times n$ matrix

What's the dimension of A?

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 6 \end{bmatrix}_{\_\_ \times \_\_}$$

General Notation

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Can write it more compactly

$$A = [a_{ij}] \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$$

Square Matrices

Square matrix: equal number of rows and columns

Example.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

Equal Matrices

Two matrices are **equal** if all their elements are identical.

Example.

$$A = \begin{bmatrix} 1 & 8 \\ 4 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 8 \\ 4 & 2 \end{bmatrix}$$

So $A = B$ if and only if $a_{ij} = b_{ij}$ for all i, j

Matrix Addition and Subtraction

How to add or take the difference between two matrices?

→ Element-by-element

→ Matrices have to have same dimension

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}$$

What is $A + B$ and $A - B$?

Scalar Multiplication

How to multiply a scalar to a matrix?

$$\lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix}$$

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}$$

What is $2B$ and $A - 2B$?

Matrix Multiplication

A whole new animal...

Only possible to multiply two matrices, $A_{m \times n}$ and $B_{p \times q}$ to get AB if $n = p$ i.e.

$$\text{number of columns in } A = \text{ number of rows in } B$$

Example. $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}_{2\times 2}$

Cannot do AB, but can do BA

Matrix Multiplication

Another example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}_{3\times 1}$$

Can we multiply A and B to find $C = AB$?

Yes, since A has 3 columns, which is equal to the number of rows in B.

Also, the dimension of C will be 2×1.

Matrix Multiplication

So how to actually multiply these matrices?

$$C = AB$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

The **element c_{ij}** is obtained by multiplying term-by-term the entries of the **ith row of A** and **jth column of B**.

Matrix Multiplication: Examples

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$$

Here,

$$C = AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}_{2\times 2}$$

Homework Questions

- Exercise 4.2: 1, 2, 4

Textbook reference: 4.1, 4.2