Constrained Optimization

ECON 441: Introduction to Mathematical Economics
Lecture 18

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Utility Maximization

Maximize:

$$f(x_1, x_2) = x_1 x_2$$

Subject to the budget constraint:

$$2x_1 + x_2 = 40$$

To find x_1^* and x_2^*, we could substitute $x_2 = 40 - 2x_1$ and use our first-order and second-order conditions.

Utility Maximization

$$f(x_1, x_2) = x_1(40 - 2x_1) = 40x_1 - 2x_1^2$$

Constrained Maximization

Substitution doesn't always work, and it gets complicated with many variables and constraints.

Thankfully, we have the method of Lagrange multipliers.

Lagrange function:

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(40 - 2x_1 - x_2)$$

λ is the Lagrange multiplier.

Now apply the first-order conditions for the case of three variables (x_1, x_2, λ).

Utility Maximization

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(40 - 2x_1 - x_2)$$

First-order conditions:

$$\frac{\partial L}{\partial x_1} = x_2 - 2\lambda = 0$$

$$\frac{\partial L}{\partial x_2} = x_1 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 40 - 2x_1 - x_2 = 0$$

Lagrange-Multiplier Method

Given an objective function

$$y = f(x_1, x_2)$$

subject to the constraint

$$g(x_1, x_2) = c$$

where c is a constant. We can write the Lagrange function as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda[c - g(x_1, x_2)]$$

Lagrange-Multiplier Method

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda[c - g(x_1, x_2)]$$

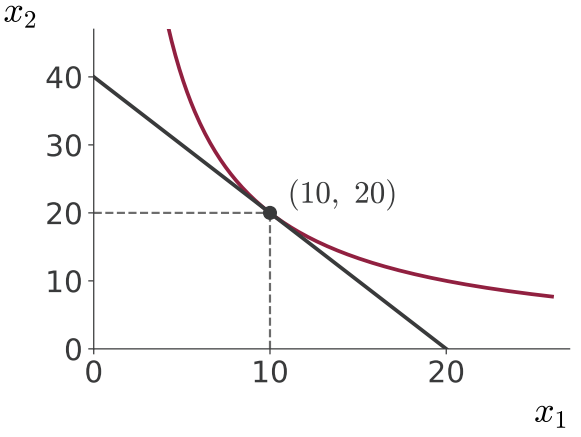
First-order conditions:

$$\frac{\partial L}{\partial x_1} = f_1(x_1^*, x_2^*) - \lambda^* g_1(x_1^*, x_2^*) = 0$$

$$\frac{\partial L}{\partial x_2} = f_2(x_1^*, x_2^*) - \lambda^* g_2(x_1^*, x_2^*) = 0$$

$$\frac{\partial L}{\partial \lambda} = c - g(x_1^*, x_2^*) = 0$$

Utility Maximization



At the optimal point:

$$\frac{\partial f/\partial x_1}{\partial f/\partial x_2} = \frac{\partial g/\partial x_1}{\partial g/\partial x_2}$$

Balance the trade-off: optimize f versus satisfy g

Utility Maximization

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(40 - 2x_1 - x_2)$$

At the optimum:

$$x_2^* = 2x_1^*, \quad 2x_1^* + x_2^* = 40 \quad \rightarrow \quad x_1^* = 10,\ x_2^* = 20,\ \lambda^* = 10$$