

Midterm 1 Review

ECON 441: Introduction to Mathematical Economics

Div Bhagia

What's on the exam

- Wednesday, September 23, in class. 70 minutes, 20 points.
- Modules I and II: Preliminaries (Lectures 1–2) and Linear Algebra (Lectures 3–7)
- Chiang & Wainwright: 2.2–2.6, p. 163, 4.1–4.7, 5.1–5.5
- The help sheet provided with the exam is on the course website
- Closed book: no phones or computers

Sets

$\mathbb{R}$ is the set of real numbers (rational and irrational)

$x \in \mathbb{R}$ denotes that x belongs to the set of real numbers

Consider the universe of all real numbers, with set A given by

$$A = \{x \mid x \geq 2\}$$

What is the complement of A ?

Set Relations and Operations

Consider the following sets:

$$A = \{x \mid x \geq 1\}$$

$$B = \{x \mid x \text{ is a positive integer}\}$$

$$C = \{x \mid 2 < x < 7\}$$

- Is $A = B$? Is $B \subset A$?
- Are C and B disjoint sets?
- What is $A \cap B$? What about $A \cup B$?
- What is $B \cap C$?

Subsets

$\emptyset$: empty or null set

What are all possible subsets of

$$S = \{a, b, c\}$$

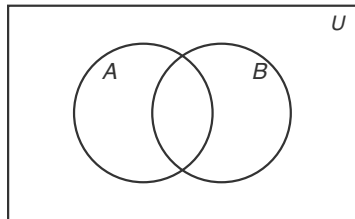
$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$

Always 2^n subsets. Here $n = 3$, so 8 subsets.

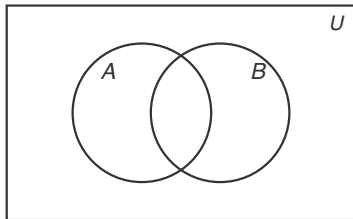
Set Operations: Venn Diagrams

Shade the following regions:

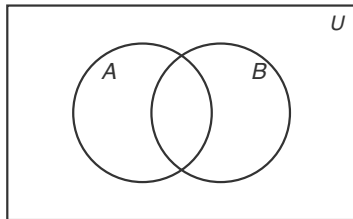
$$A \cup B$$



$$A \cap B$$



$$A^c$$



Cartesian Product

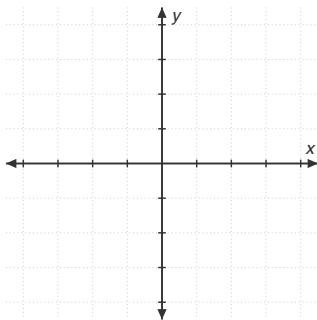
The **Cartesian product** of two sets A and B is the set of all ordered pairs whose first element comes from A and second from B :

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Example. $A = \{0, 1\}$, $B = \{5, 6, 7\}$. What is $A \times B$? How many pairs?

Cartesian Plane

$$\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$$



Can have $\mathbb{R}^3, \mathbb{R}^4, \dots, \mathbb{R}^n$

Relations and Functions

Relation: subset of the Cartesian product, e.g. $\{(x, y) \mid y \leq x\}$

Function: a relation where for each x there is a unique y

$$f : X \rightarrow Y, \quad y = f(x)$$

X : domain, Y : codomain, $f(X)$: range

Is $y = |x|$ a function? Is $\{(x, y) \mid x = y^2\}$ a function of x ?

Inverse of a Function

Function $y = f(x)$ has an inverse if it is a one-to-one mapping, i.e. each value of y is associated with a unique value of x .

The inverse function

$$x = f^{-1}(y)$$

returns the corresponding value of x for each y .

One-to-one mapping is unique to strictly monotonic functions.

Find the inverse of $y = 4x + 8$. Does $y = x^2$ have an inverse?

Summation Notation

$$\sum_{i=1}^N x_i = x_1 + x_2 + \dots + x_N$$

Let $x_1 = 3$, $x_2 = 5$, $x_3 = 1$, $x_4 = 7$.

- Evaluate $\sum_{i=2}^4 x_i$
- Evaluate $\sum_{i=1}^4 (2x_i + 3)$
- Expand $\sum_{i=1}^3 b_i x^i$

Necessary vs. Sufficient Conditions

q is a **necessary** condition for p if $p \implies q$

q is a **sufficient** condition for p if $q \implies p$

If both hold, $p \iff q$ and each is **necessary and sufficient** for the other

Example. p : you got an A in this course; q : you passed this course.

Is passing **necessary** for an A? Is it **sufficient**?

Matrices

A matrix with m rows and n columns is an $m \times n$ matrix

$$A = [a_{ij}] \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n$$

Square matrix: equal number of rows and columns

Two matrices are **equal** if all their elements are identical

What is the dimension of $A = \begin{bmatrix} 5 & 0 \\ 2 & 7 \\ 1 & 3 \end{bmatrix}$?

Addition and Scalar Multiplication

Element-by-element, and the matrices must have the same dimension:

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 0 \\ -1 & 6 \end{bmatrix}$$

Find $A + B$, $A - B$, and $3A$.

Matrix Multiplication

Only possible to multiply $A_{m \times n}$ and $B_{p \times q}$ to get AB if $n = p$:

number of columns in A = number of rows in B

$$C = AB, \quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj}$$

The element c_{ij} comes from the i th row of A and the j th column of B .

Matrix Multiplication

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}_{3 \times 1}$$

Is AB defined? What is its dimension? Find it.

What about BA ?

Vectors and Linear Dependence

Matrices with only one column are **column vectors**; with only one row, **row vectors**

A set of vectors is **linearly dependent** if and only if any one of them can be expressed as a linear combination of the remaining vectors

Are these linearly independent?

$$v_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Special Matrices

Identity matrix I : 1s on the principal diagonal, 0s elsewhere; acts like 1, so $AI = IA = A$

Null matrix: all elements 0; $A + 0 = A$ and $A0 = 0$

Idempotent: A is idempotent if $A = A^k$, i.e. it is unchanged when multiplied by itself

Is an identity matrix idempotent?

Transpose of a Matrix

A^T : interchange rows and columns

Find A^T for

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 5 & 3 \end{bmatrix}$$

What is the dimension of A^T ? What about $A^T A$?

Inverse of a Matrix

For a **square** matrix A , its inverse A^{-1} is defined as

$$AA^{-1} = A^{-1}A = I$$

If the inverse exists, A is **nonsingular**

Squareness is **necessary** but not **sufficient**

Conditions for Nonsingularity

Sufficient condition: **rows (or equivalently) columns are linearly independent**

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$$

Which is singular, and which is nonsingular?

Determinant

The determinant $|A|$ is a unique scalar associated with a **square** matrix A

For a 2×2 matrix:

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

A **minor** $|M_{ij}|$ deletes the i th row and j th column; the cofactor is

$$|C_{ij}| = (-1)^{i+j} |M_{ij}|$$

Then, expanding along any row or column:

$$|A| = \sum_{j=1}^n a_{ij} |C_{ij}|$$

Find the Determinant

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Is A nonsingular?

Criteria for Nonsingularity

The following statements are equivalent:

- $|A| \neq 0$
- Rows (or equivalently columns) of A are independent
- A is nonsingular
- A^{-1} exists
- A unique solution to $Ax = b$ ($x^* = A^{-1}b$) exists

Matrix Inversion

Adjoint of a nonsingular $n \times n$ matrix:

$$\text{adj}A = C^T = \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \vdots & \vdots & \cdots & \vdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix}$$

The **inverse** of A is defined as:

$$A^{-1} = \frac{1}{|A|} \text{adj}A$$

Find the Inverse

Example 1

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Example 2

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Solving a System of Equations

By matrix inversion: premultiply $Ax = b$ by A^{-1} to get

$$x^* = A^{-1}b$$

By Cramer's rule: form A_k by replacing the k th column of A with b ; then

$$x_k^* = \frac{|A_k|}{|A|}$$

If A is singular, a unique solution does not exist.

A Simple Economic Model

National income Y and consumption C , where investment and government spending together are 30:

$$Y = C + 30 \quad C = 20 + \frac{1}{2}Y$$

Rearranged, this is

$$Y - C = 30$$

$$-Y + 2C = 40$$

Write it as $Ax = b$. Solve it two ways: by matrix inversion, and by Cramer's rule.

For your review

- Lecture slides and worksheets
- Practice problems 1–7 with solutions, also collected as one PDF per module
- Module notes for Linear Algebra
- Videos on finding the inverse, under Lecture 6
- Sample exam and help sheet, on the Exams page
- Additional problems in the textbook

Good luck!