

Solutions: Calculus

ECON 441: Introduction to Mathematical Economics

Instructor: Div Bhagia

Exercise 6.2

2. $y = 5x^2 - 4x$

(a)

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \frac{5(x + \Delta x)^2 - 4(x + \Delta x) - (5x^2 - 4x)}{\Delta x} \\ &= \frac{5(x^2 + 2x\Delta x + (\Delta x)^2) - 4x - 4\Delta x - 5x^2 + 4x}{\Delta x} \\ &= \frac{10x\Delta x + 5(\Delta x)^2 - 4\Delta x}{\Delta x} = 10x - 4 + 5\Delta x\end{aligned}$$

(b)

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} (10x - 4 + 5\Delta x) = 10x - 4$$

(c) $f'(2) = 10 \times 2 - 4 = 16$ and $f'(3) = 10 \times 3 - 4 = 26$.

3. $y = 5x - 2$

(a)

$$\frac{\Delta y}{\Delta x} = \frac{5(x + \Delta x) - 2 - (5x - 2)}{\Delta x} = \frac{5\Delta x}{\Delta x} = 5$$

It is a constant function.

(b) No: the difference quotient equals 5 whatever the size of Δx , so its limit as $\Delta x \rightarrow 0$ is also 5, i.e. $dy/dx = 5$.

Exercise 7.1

3. (a) $f'(x) = 18$, so $f'(1) = 18$ and $f'(2) = 18$.

(b) $f'(x) = 3cx^2$, so $f'(1) = 3c$ and $f'(2) = 12c$.

(c) $f'(x) = 10x^{-3}$, so $f'(1) = 10$ and $f'(2) = \frac{10}{8} = \frac{5}{4}$.

(d) $f'(x) = x^{1/3} = \sqrt[3]{x}$, so $f'(1) = 1$ and $f'(2) = \sqrt[3]{2}$.

(e) $f'(w) = 2w^{-2/3}$, so $f'(1) = 2$ and $f'(2) = 2 \cdot 2^{-2/3} = 2^{1/3}$.

(f) $f'(w) = \frac{1}{2}w^{-7/6}$, so $f'(1) = \frac{1}{2}$ and $f'(2) = \frac{1}{2} \cdot 2^{-7/6} = 2^{-13/6}$.

Exercise 7.2

3. (d) Let $f(x) = ax - b$ and $g(x) = cx^2$, so $f'(x) = a$ and $g'(x) = 2cx$.

$$\begin{aligned}\frac{d}{dx} [f(x)g(x)] &= f'(x)g(x) + f(x)g'(x) \\ &= acx^2 + (ax - b)2cx \\ &= acx^2 + 2acx^2 - 2bcx \\ &= 3acx^2 - 2bcx\end{aligned}$$

(e) Let $f(x) = (2 - 3x)(1 + x)$ and $g(x) = x + 2$, so $g'(x) = 1$. Using the product rule on $f(x)$ itself:

$$f'(x) = -3(1 + x) + (2 - 3x)(1) = -(6x + 1)$$

Then

$$\begin{aligned}\frac{d}{dx} [f(x)g(x)] &= f'(x)g(x) + f(x)g'(x) \\ &= -(6x + 1)(x + 2) + (2 - 3x)(1 + x) \\ &= -6x^2 - 13x - 2 + 2 - x - 3x^2 \\ &= -9x^2 - 14x = -x(9x + 14)\end{aligned}$$

7. (a) With $f(x) = x^2 + 3$, $f'(x) = 2x$, and $g(x) = x$, $g'(x) = 1$:

$$\begin{aligned}\frac{d}{dx} \frac{f(x)}{g(x)} &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{(2x)x - (x^2 + 3) \cdot 1}{x^2} = \frac{x^2 - 3}{x^2}\end{aligned}$$

(b)

$$\frac{d}{dx} \frac{x + 9}{x} = \frac{1 \cdot x - 1 \cdot (x + 9)}{x^2} = \frac{-9}{x^2}$$

(c)

$$\frac{d}{dx} \frac{6x}{x + 5} = \frac{6(x + 5) - 1 \cdot 6x}{(x + 5)^2} = \frac{30}{(x + 5)^2}$$

(d)

$$\begin{aligned}
 \frac{d}{dx} \frac{ax^2 + b}{cx + d} &= \frac{2ax(cx + d) - (ax^2 + b)c}{(cx + d)^2} \\
 &= \frac{2acx^2 + 2adx - acx^2 - bc}{(cx + d)^2} \\
 &= \frac{acx^2 + 2adx - bc}{(cx + d)^2}
 \end{aligned}$$

8. $f(x) = ax + b$

(a) $f'(x) = a$

(b) $\frac{d}{dx}(ax^2 + bx) = 2ax + b$

(c)

$$\frac{d}{dx} \frac{1}{ax + b} = \frac{0 \cdot (ax + b) - a \cdot 1}{(ax + b)^2} = \frac{-a}{(ax + b)^2}$$

(d)

$$\frac{d}{dx} \frac{ax + b}{x} = \frac{ax - (ax + b) \cdot 1}{x^2} = \frac{-b}{x^2}$$

Exercise 7.3

1.

$$\frac{dy}{du} = 3u^2 + 2 \quad \frac{du}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (3u^2 + 2)(-2x) = -2x \left[3(5 - x^2)^2 + 2 \right]$$

2.

$$\begin{aligned}
 \frac{dw}{dx} &= \frac{dw}{dy} \cdot \frac{dy}{dx} \\
 &= 2ay(2bx + c) \\
 &= 2a(bx^2 + cx)(2bx + c)
 \end{aligned}$$

3. (a) Let $z = 3x^2 - 13$, so $y = z^3$:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 3z^2(6x) = 18x(3x^2 - 13)^2$$

(b) Let $z = 7x^3 - 5$, so $y = z^9$:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 9z^8 (21x^2) = 189x^2 (7x^3 - 5)^8$$

(c) Let $z = ax + b$, so $y = z^5$:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 5z^4(a) = 5a(ax + b)^4$$

4. Chain rule, with $z = 16x + 3$ and $y = z^{-2}$:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = -2z^{-3}(16) = \frac{-32}{(16x + 3)^3}$$

Quotient rule, where the derivative of the denominator is $2(16x+3) \cdot 16 = 32(16x+3)$:

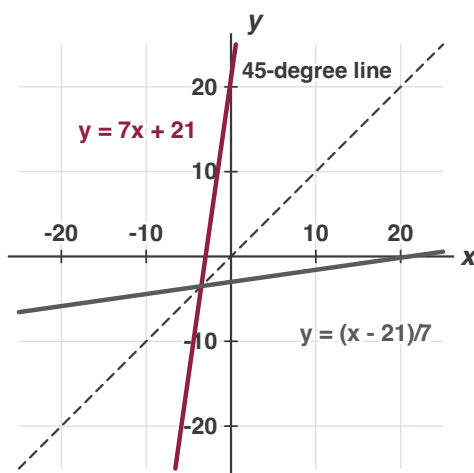
$$\frac{dy}{dx} = \frac{0 \cdot (16x + 3)^2 - 1 \cdot 32(16x + 3)}{(16x + 3)^4} = \frac{-32}{(16x + 3)^3}$$

The answers are identical.

5. The inverse function is $x = \frac{y-21}{7}$.

$$\frac{dy}{dx} = 7 \quad \frac{dx}{dy} = \frac{1}{7} = \frac{1}{dy/dx}$$

so the inverse-function rule holds. To graph the inverse with its own independent variable on the horizontal axis, write it as $y = (x - 21)/7$. The two lines are mirror images of each other across the 45-degree line $y = x$:



6. (a)

$$\frac{dy}{dx} = -6x^5 < 0 \text{ for } x > 0$$

So the function is strictly decreasing, hence strictly monotonic, and

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{-1}{6x^5}$$

(b)

$$\frac{dy}{dx} = 20x^4 + 3x^2 + 3 > 0 \text{ for all } x$$

So the function is strictly increasing, hence strictly monotonic, and

$$\frac{dx}{dy} = \frac{1}{20x^4 + 3x^2 + 3}$$

Exercise 10.5

1. (a) $\frac{dy}{dt} = 2e^{2t+4}$

(b) $\frac{dy}{dt} = -9e^{1-9t}$

(c) $\frac{dy}{dt} = 2te^{t^2+1}$

(d) $\frac{dy}{dt} = -10te^{2-t^2}$

- (e) $\frac{dy}{dx} = (2ax + b)e^{ax^2+bx+c}$
- (f) $\frac{dy}{dx} = x \frac{d}{dx} e^x + e^x \frac{d}{dx} x = xe^x + e^x = (x+1)e^x$
- (g) $\frac{dy}{dx} = x^2 (2e^{2x}) + 2xe^{2x} = 2x(x+1)e^{2x}$
- (h) $\frac{dy}{dx} = a (xbe^{bx+c} + e^{bx+c}) = a(bx+1)e^{bx+c}$
3. (a) $\frac{dy}{dt} = \frac{35t^4}{7t^5} = \frac{5}{t}$
- (b) $\frac{dy}{dt} = \frac{10at^9}{at^{10}} = \frac{10}{t}$
- (c) $\frac{dy}{dt} = \frac{1}{t+19}$
- (d) $\frac{dy}{dt} = 5 \cdot \frac{2(t+1)}{(t+1)^2} = \frac{10}{t+1}$
- (e) $\frac{dy}{dx} = \frac{1}{x} - \frac{1}{1+x} = \frac{1}{x(1+x)}$
- (f) $\frac{dy}{dx} = \frac{d}{dx} [\ln x + 8 \ln(1-x)] = \frac{1}{x} - \frac{8}{1-x} = \frac{1-9x}{x(1-x)}$
- (g) $\frac{dy}{dx} = \frac{d}{dx} [\ln 2x - \ln(1+x)] = \frac{2}{2x} - \frac{1}{1+x} = \frac{1}{x(1+x)}$
- (h) $\frac{dy}{dx} = 5x^4 \cdot \frac{2x}{x^2} + 20x^3 \ln x^2 = 10x^3 (1 + 2 \ln x^2) = 10x^3 (1 + 4 \ln x)$
7. (a) Since $\ln y = \ln 3x - \ln(x+2) - \ln(x+4)$,

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} - \frac{1}{x+2} - \frac{1}{x+4} = \frac{8-x^2}{x(x+2)(x+4)}$$

Hence,

$$\frac{dy}{dx} = \frac{8-x^2}{x(x+2)(x+4)} \cdot \frac{3x}{(x+2)(x+4)} = \frac{3(8-x^2)}{(x+2)^2(x+4)^2}$$

- (b) Since $\ln y = \ln(x^2+3) + x^2 + 1$,

$$\frac{1}{y} \frac{dy}{dx} = \frac{2x}{x^2+3} + 2x = \frac{2x(x^2+4)}{x^2+3}$$

Hence,

$$\frac{dy}{dx} = \frac{2x(x^2+4)}{x^2+3} \cdot (x^2+3)e^{x^2+1} = 2x(x^2+4)e^{x^2+1}$$

Exercise 7.4

1. (a)

$$\frac{\partial y}{\partial x_1} = 6x_1^2 - 22x_1x_2 \quad \frac{\partial y}{\partial x_2} = -11x_1^2 + 6x_2$$

(d)

$$\frac{\partial y}{\partial x_1} = \frac{5(x_2 - 2) - (5x_1 + 3) \cdot 0}{(x_2 - 2)^2} = \frac{5}{x_2 - 2}$$

$$\frac{\partial y}{\partial x_2} = \frac{0 \cdot (x_2 - 2) - 1 \cdot (5x_1 + 3)}{(x_2 - 2)^2} = \frac{-(5x_1 + 3)}{(x_2 - 2)^2}$$

2. (a)

$$f_x = 2x + 5y \quad f_y = 5x - 3y^2$$

(b)

$$f_x = (2x)(x - 2) + (x^2 - 3y) \cdot 1 = 3x^2 - 4x - 3y$$

$$f_y = -3(x - 2) + (x^2 - 3y) \cdot 0 = -3x + 6$$

3. For (a), $f_x(1, 2) = 2(1) + 5(2) = 12$.For (b), $f_x(1, 2) = 3(1)^2 - 4(1) - 3(2) = -7$.

5. (a)

$$U_1 = \frac{\partial U}{\partial x_1} = 2(x_1 + 2)(x_2 + 3)^3$$

$$U_2 = \frac{\partial U}{\partial x_2} = 3(x_1 + 2)^2(x_2 + 3)^2$$

(b)

$$U_1(3, 3) = 2(3 + 2)(3 + 3)^3 = 2 \times 5 \times 216 = 2160$$

7. (a)

$$f_x = 2x \quad f_y = 3y^2 \quad f_z = 4z^3 \quad \nabla f(x, y, z) = \begin{bmatrix} 2x \\ 3y^2 \\ 4z^3 \end{bmatrix}$$

(b)

$$f_x = yz \quad f_y = xz \quad f_z = xy \quad \nabla f(x, y, z) = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$$

Exercise 8.1

1. (a) Since $y = -x^3 - 3x$,

$$dy = (-3x^2 - 3) dx = -3(x^2 + 1) dx$$

4. (a) Writing $Q = kP^{-n}$, $\frac{dQ}{dP} = -nkP^{-n-1}$, so

$$\varepsilon_d = \frac{dQ}{dP} \cdot \frac{P}{Q} = \frac{-nkP^{-n-1} \cdot P}{kP^{-n}} = -n$$

No: the elasticity is the constant $-n$, whatever the price.

- (b) With $n = 1$, $Q = k/P$, so $PQ = k$: the demand curve is a rectangular hyperbola, downward sloping and approaching both axes. The point elasticity is $\varepsilon_d = -1$ everywhere (unit elastic).

6. (a) At $P = 20$ and $Y = 5,000$, $Q = 100 - 40 + 100 = 160$, so

$$\varepsilon_P = \frac{\partial Q}{\partial P} \cdot \frac{P}{Q} = -2 \cdot \frac{20}{160} = -0.25$$

- (b)

$$\varepsilon_Y = \frac{\partial Q}{\partial Y} \cdot \frac{Y}{Q} = 0.02 \cdot \frac{5000}{160} = 0.625$$

Exercise 8.2

3. (a)

$$\frac{\partial y}{\partial x_1} = \frac{1 \cdot (x_1 + x_2) - x_1 \cdot 1}{(x_1 + x_2)^2} = \frac{x_2}{(x_1 + x_2)^2}$$

$$\frac{\partial y}{\partial x_2} = \frac{-x_1}{(x_1 + x_2)^2}$$

$$dy = \frac{x_2 dx_1 - x_1 dx_2}{(x_1 + x_2)^2}$$

- 4.

$$\varepsilon_{QP} = \frac{\partial Q}{\partial P} \cdot \frac{P}{Q} = \frac{2bP \cdot P}{Q} = \frac{2bP^2}{a + bP^2 + R^{1/2}}$$

$$\varepsilon_{QR} = \frac{\partial Q}{\partial R} \cdot \frac{R}{Q} = \frac{1}{2}R^{-1/2} \cdot \frac{R}{Q} = \frac{R^{1/2}}{2(a + bP^2 + R^{1/2})}$$

5. Write $Q = a + bP^2 + R^{1/2}$ and assume $Q > 0$. For $\varepsilon_{QP} = 2bP^2/Q$:

$$\frac{\partial \varepsilon_{QP}}{\partial P} = \frac{4bPQ - 2bP^2 \cdot 2bP}{Q^2} = \frac{4bP(a + R^{1/2})}{Q^2}$$

$$\frac{\partial \varepsilon_{QP}}{\partial R} = \frac{-bP^2 R^{-1/2}}{Q^2} < 0$$

So ε_{QP} is strictly decreasing in R . In P it is strictly increasing if $a + R^{1/2} > 0$ and strictly decreasing if $a + R^{1/2} < 0$: the sign does not depend on P , so it is strictly monotonic in P either way.

For $\varepsilon_{QR} = R^{1/2}/(2Q)$:

$$\frac{\partial \varepsilon_{QR}}{\partial P} = \frac{-bP R^{1/2}}{Q^2} < 0$$

$$\frac{\partial \varepsilon_{QR}}{\partial R} = \frac{R^{-1/2}Q - 1}{4Q^2} = \frac{R^{-1/2}(a + bP^2)}{4Q^2}$$

So ε_{QR} is strictly decreasing in P . In R it is strictly increasing if $a + bP^2 > 0$ and strictly decreasing if $a + bP^2 < 0$: again the sign does not depend on R , so it is strictly monotonic in R .

6.

$$\varepsilon_{XP} = \frac{\partial X}{\partial P} \cdot \frac{P}{X} = \frac{-2P^{-3} \cdot P}{Y_f^{1/2} + P^{-2}} = \frac{-2}{Y_f^{1/2} P^2 + 1} < 0$$

7. (b)

$$\begin{aligned} dU &= U_x dx + U_y dy \\ &= 14xy^3 dx + 21x^2y^2 dy \\ &= 7xy^2 (2y dx + 3x dy) \end{aligned}$$

(f)

$$\begin{aligned} dU &= 3(x - 3y)^2 dx + 3(x - 3y)^2 (-3) dy \\ &= 3(x - 3y)^2 (dx - 3 dy) \end{aligned}$$

Exercise 8.4

2. (a)

$$\begin{aligned}
 \frac{dz}{dt} &= z_x \frac{dx}{dt} + z_y \frac{dy}{dt} \\
 &= (2x - 8y)(3) + (-8x - 3y^2)(-1) \\
 &= 14x - 24y + 3y^2 \\
 &= 42t - 24(1 - t) + 3(1 - t)^2 = 3t^2 + 60t - 21
 \end{aligned}$$

(b)

$$\begin{aligned}
 \frac{dz}{dt} &= z_u \frac{du}{dt} + z_v \frac{dv}{dt} + z_t \\
 &= 7(4t) + t \cdot 1 + v \\
 &= 28t + t + t + 1 = 30t + 1
 \end{aligned}$$

(c)

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_t = b f_x + k f_y + f_t$$

4. (a)

$$\begin{aligned}
 \frac{\$W}{\$u} &= W_x \frac{\partial x}{\partial u} + W_y \frac{\partial y}{\partial u} + W_u \\
 &= (2ax + by)\alpha + bx\gamma + c \\
 &= (2a\alpha + b\gamma)(\alpha u + \beta v) + \alpha b\gamma u + c \\
 \frac{\$W}{\$v} &= W_x \frac{\partial x}{\partial v} + W_y \frac{\partial y}{\partial v} \\
 &= (2ax + by)\beta + bx \cdot 0 \\
 &= \beta [(2a\alpha + b\gamma)u + 2a\beta v]
 \end{aligned}$$

(b)

$$\begin{aligned}
 \frac{\$W}{\$u} &= f_1 \frac{\partial x_1}{\partial u} + f_2 \frac{\partial x_2}{\partial u} = 10u f_1 + f_2 \\
 \frac{\$W}{\$v} &= f_1 \frac{\partial x_1}{\partial v} + f_2 \frac{\partial x_2}{\partial v} = 3f_1 - 12v^2 f_2
 \end{aligned}$$

Exercise 8.51. (a) With $F_x = -6$ and $F_y = 1$, the implicit-function rule gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-6}{1} = 6$$

Equivalently, solving for $y = 6x - 7$ and differentiating gives $dy/dx = 6$.

(b) With $F_x = 12$ and $F_y = 3$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{12}{3} = -4$$

(c) With $F_x = 2x + 6$ and $F_y = -1$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+6}{-1} = 2x+6$$

2. (d)

$$F_x = 18x^2 \quad F_y = -3$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{18x^2}{-3} = 6x^2$$

3. (a)

$$F_x = 2xy^3 + yz \quad F_y = 3x^2y^2 + xz \quad F_z = 2z + xy$$

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2xy^3 + yz}{3x^2y^2 + xz}$$

$$\frac{\partial y}{\partial z} = -\frac{F_z}{F_y} = -\frac{2z + xy}{3x^2y^2 + xz}$$

Exercise 14.2

1. (a)

$$\int 16x^{-3} dx = 16 \cdot \frac{x^{-2}}{-2} + c = -8x^{-2} + c$$

(c)

$$\int (x^5 - 3x) dx = \int x^5 dx - 3 \int x dx = \frac{x^6}{6} - \frac{3x^2}{2} + c$$

(d)

$$\int 2e^{-2x} dx = 2 \int e^{-2x} dx = 2 \cdot \frac{e^{-2x}}{-2} + c = -e^{-2x} + c$$

Exercise 14.3

1. (a)

$$\int_1^3 \frac{1}{2}x^2 dx = \left[\frac{x^3}{6} \right]_1^3 = \frac{27-1}{6} = \frac{13}{3}$$

(e)

$$\left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx \right]_{-1}^1 = \left(\frac{a}{3} + \frac{b}{2} + c \right) - \left(-\frac{a}{3} + \frac{b}{2} - c \right) = \frac{2a}{3} + 2c$$

2. (a)

$$\int_1^2 e^{-2x} dx = \left[\frac{e^{-2x}}{-2} \right]_1^2 = -\frac{1}{2} (e^{-4} - e^{-2}) = \frac{1}{2} (e^{-2} - e^{-4})$$

(d)

$$\left[\ln x + \ln(1+x) \right]_e^6 = \ln 6 + \ln 7 - \ln e - \ln(1+e) = \ln 42 - 1 - \ln(1+e)$$

5. (a)

$$\int_0^b \frac{c}{b} dx = \left[\frac{cx}{b} \right]_0^b = \frac{cb}{b} - 0 = c$$

(b)

$$\int_0^c 1 dt = \left[t \right]_0^c = c - 0 = c$$