Midterm Sample: Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

1. (5 pts)

   (a) (1 pt) *Is f a valid function?* **Yes**: a function may map two values of x to the same value of y; what it cannot do is map one x to two values of y.

   (b) (1 pt) $A \cup B =$ Set of all integers.

   (c) (1 pt)

   $$\sum_{j=1}^{2}\sum_{i=1}^{2} X_i Y_j = \sum_{j=1}^{2}\left(X_1 Y_j + X_2 Y_j\right) = X_1 Y_1 + X_2 Y_1 + X_1 Y_2 + X_2 Y_2$$

   (d) (1 pt) No: since the function is not one-to-one (it is not monotonic), the inverse does not exist.

   (e) (1 pt) $(0, 2)$

2. (5 pts) $A = I - X(X^T X)^{-1} X^T$

   (a) (3 pts) Say the dimension of X is $m \times n$. Then the dimension of $X^T_{n\times m} X_{m\times n}$ is $n \times n$, so the dimension of $(X^T X)^{-1}$ is also $n \times n$. This implies that the dimension of $X_{m\times n}(X^T X)^{-1}_{n\times n} X^T_{n\times m}$ is $m \times m$. Hence A is square (as is $X^T X$), but X need not be square.

   (b) (2 pts)

   $$\begin{aligned} AA &= (I - X(X^T X)^{-1} X^T)(I - X(X^T X)^{-1} X^T) \\ &= I - X(X^T X)^{-1} X^T - X(X^T X)^{-1} X^T + X \underbrace{(X^T X)^{-1} X^T X}_{I} (X^T X)^{-1} X^T \\ &= I - X(X^T X)^{-1} X^T = A \end{aligned}$$

3. (10 pts)

(a) (2 pts) Setting supply equal to demand in each market:

$$20 - p_b + p_t = 12 + p_b \implies 2p_b - p_t = 8$$

$$36 + p_b - p_t + p_a = 30 + p_t \implies -p_b + 2p_t - p_a = 6$$

$$20 + p_t - p_a = 16 + p_a \implies -p_t + 2p_a = 4$$

(b) (1 pt)

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad v = \begin{bmatrix} p_b \\ p_t \\ p_a \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$$

(c) (2 pts) We first need to calculate all the cofactors of A.

$$|C_{11}| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \quad |C_{12}| = -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2 \quad |C_{13}| = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$$

$$|C_{21}| = -\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = 2 \quad |C_{22}| = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \quad |C_{23}| = -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = 2$$

$$|C_{31}| = \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1 \quad |C_{32}| = -\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = 2 \quad |C_{33}| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$adj A = C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(d) (1 pt)

$$|A| = a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}| = 2 \times 3 + (-1) \times 2 + 0 \times 1 = 4$$

A is nonsingular as $|A| \neq 0$.

(e) (1 pt) Premultiplying by A^{-1}: $A^{-1}Av = A^{-1}b$. Since $A^{-1}A = I$, we have $v^* = A^{-1}b$.

(f) (3 pts) Since $A^{-1} = \frac{1}{|A|} adjA$,

$$v^* = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 24+12+4 \\ 16+24+8 \\ 8+12+12 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 8 \end{bmatrix}$$

So $p_b^* = 10$, $p_t^* = 12$, and $p_a^* = 8$. (Substituting back: $q_b^* = 22$, $q_t^* = 42$, $q_a^* = 24$ in both the demand and supply equations.)