Optimization Examples

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Example 1: Utility Maximization

Consider the following utility maximization problem:

$$\max_{\{x_1,x_2\}} \quad x_1^{\alpha}x_2^{\beta} \quad s.t. \quad p_1x_1 + p_2x_2 = m$$

Here, p_1 and p_2 are the prices of goods 1 and 2, and m is the total income available to spend on the two goods. To find the critical points, we start by setting up the Lagrange function:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{\alpha}x_2^{\beta} + \lambda(m - p_1x_1 - p_2x_2)$$

The first-order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha x_1^{\alpha-1}x_2^{\beta} - \lambda p_1 = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \beta x_1^{\alpha}x_2^{\beta-1} - \lambda p_2 = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1x_1 - p_2x_2 = 0 \tag{3}$$

From equations (1) and (2), we have $\alpha x_1^{\alpha-1}x_2^{\beta} = \lambda p_1$ and $\beta x_1^{\alpha}x_2^{\beta-1} = \lambda p_2$, dividing the first expression by the second, we get:

$$\frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{\beta x_1^{\alpha}x_2^{\beta-1}} = \frac{\lambda p_1}{\lambda p_2}$$

Simplifying this expression further:

$$\frac{\alpha x_2^{\beta} x_2^{1-\beta}}{\beta x_1^{\alpha} x_1^{1-\alpha}} = \frac{p_1}{p_2} \rightarrow \frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2}$$

Then we can write that,

$$x_2 = \frac{\beta}{\alpha} \cdot \frac{p_1}{p_2} \cdot x_1$$

Plugging the above expression for x_2 in equation (3):

$$m - p_1 x_1 - p_2 x_2 = m - p_1 x_1 - \frac{\beta}{\alpha} \cdot p_1 x_1 = m - p_1 x_1 \left(1 + \frac{\beta}{\alpha}\right) = 0$$

So we can find,

$$x_1^* = \frac{\alpha}{\alpha + \beta} \frac{m}{p_1}$$

Plugging back x_1 in the expression for x_2, we can find:

$$x_2^* = \frac{\beta}{\alpha + \beta} \frac{m}{p_2}$$

Example 2: Cost Minimization

Consider the following cost minimization problem where p is the price of capital, and w is the price of labor. Quantity is constrained at $\bar{Q}$.

$$\max_{\{K,L\}} \quad pK + wL \quad s.t. \quad Q(K,L) = \bar{Q}$$

Lagrange function:

$$\mathcal{L}(x_1, x_2, \lambda) = pK + wL + \lambda(\bar{Q} - Q(K,L))$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial K} = p - \lambda Q_K(K, L) = 0$$
$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda Q_L(K, L) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{Q} - Q(K, L)$$

Note that here $Q_K = \frac{\partial Q}{\partial K}$ and $Q_L = \frac{\partial Q}{\partial L}$.

So optimal K and L satisfy:

$$(1). \quad \frac{Q_K(K^*, L^*)}{Q_L(K^*, L^*)} = \frac{p}{w}, \qquad (2). \quad Q(K^*, L^*) = \bar{Q}$$

Example 3: Inter-Temporal Utility Maximization

You are given the following inter-temporal utility function:

$$U = U(c_1, c_2) = \ln c_1 + \beta \ln c_2 \tag{4}$$

where c_1 and c_2 is consumption in period 1 and 2, respectively. $0 < \beta < 1$ is the rate at which you discount the future and it measures your impatient. You earn income $y_1 > 0$ in period 1 and income $y_2 > 0$ in period 2. Any of the income you save s in period 1 earns interest $r > 0$. So,

$$c_1 + s = y_1, \qquad c_2 = y_2 + (1 + r)s$$

Combining these constraints:

$$c_1 + \frac{1}{1+r} c_2 = y_1 + \frac{1}{1+r} y_2$$

Let the present-discounted income be denoted by m, such that:

$$m = y_1 + \frac{1}{1+r} y_2$$

You want to choose c_1 and c_2 to maximize utility $U(c_1, c_2)$ in equation (4) subject to the constraint:

$$c_1 + \frac{1}{1+r}c_2 = m \tag{5}$$

Lagrangian function:

$$L(c_1, c_2, \lambda) = \ln c_1 + \beta \ln c_2 + \lambda \left(m - c_1 - \frac{1}{1+r}c_2 \right)$$

First order conditions:

$$\frac{\partial L}{\partial c_1} = \frac{1}{c_1^*} - \lambda^* = 0 \tag{6}$$

$$\frac{\partial L}{\partial c_2} = \frac{\beta}{c_2^*} - \frac{\lambda^*}{1+r} = 0 \tag{7}$$

$$\frac{\partial L}{\partial \lambda} = m - c_1^* - \frac{1}{1+r}c_2^* = 0 \tag{8}$$

Note that from (6), $\lambda^* = 1/c_1^*$, plugging this in (7), we get:

$$c_2^* = \beta(1+r)c_1^*$$

Plugging this expression for c_2^* in (8):

$$c_1^* + \frac{1}{1+r}\beta(1+r)c_1^* = m \rightarrow c_1^* = \frac{m}{1+\beta}$$

In which case,

$$c_2^* = \beta(1+r)c_1^* = \frac{\beta m(1+r)}{1+\beta}$$

Finally, since $\lambda^* = 1/c_1^*$,

$$\lambda^* = \frac{1}{c_1^*} = \frac{1+\beta}{m}$$

Now suppose we are interested in knowing how utility changes due to changes in total income. By the envelope theorem:

$$\frac{\partial U^*}{\partial m} = \frac{\partial L^*}{\partial m} = \lambda^* = \frac{1+\beta}{m}$$

Similarly, how utility changes due to a change in interest rate would be given by:

$$\frac{\partial U^*}{\partial r} = \frac{\partial L^*}{\partial r} = \lambda^* \cdot \frac{1}{(1+r)^2} \cdot c_2^* = \frac{\beta}{1+r}$$