Homework 9 Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Exercise 11.2

1. $z = x^2 + xy + 2y^2 + 3$

   F.0.C:

   $f_x = 2x + y = 0$

   $f_y = x + 4y = 0$

   To solve the above system, plug in $x = -4y$ in the first equation.

   $$2x + y = -8y + y = 0$$

   Critical point: $(0, 0)$

   S.0.C:

   $f_{xx} = 2 > 0$

   $f_{yy} = 4 > 0$

   $f_{xy} = 1$

   $f_{xx} f_{yy} = 8 > 1 = f_{xy}^2$

   f has a local minimum at $(0, 0)$.

2. $z = -x^2 - y^2 + 6x + 2y$

   F.0.C:

   $f_x = -2x + 6 = 0$

   $f_y = -2y + 2 = 0$

Critical point: $(3, 1)$

S.0.C:

$$
\begin{aligned}
f_{xx} &= -2 < 0 \\
f_{yy} &= -2 < 0 \\
f_{xy} &= 0 \\
f_{xx}f_{yy} &= 4 > 0 = f_{xy}^2
\end{aligned}
$$

f has a local maximum at $(3, 1)$.

3. $z = ax^2 + by^2 + c$

   FOC :

$$
\begin{aligned}
f_x &= 2ax = 0 \\
f_y &= 2by = 0
\end{aligned}
$$

   Critical point : $(0, 0)$

   SOC :

$$
\begin{aligned}
f_{xx} &= 2a \\
f_{yy} &= 2b \\
f_{xy} &= 0 \\
f_{xx}f_{yy} &= 4ab
\end{aligned}
$$

   (a) $a > 0, b > 0$

   $f_{xx} > 0, f_{yy} > 0$

   $f_{xx}f_{yy} = 4ab > 0 = f_{xy}^2$

   Local minimum.

   (b) $a < 0, b < 0$

   $f_{xx} < 0, f_{yy} < 0$

   $f_{xx}f_{yy} = 4ab > 0 = f_{xy}^2$

Local maximum.

(c) $a > 0, \quad b < 0$

$f_{xx} > 0, f_{yy} < 0$

Neither maximum nor minimum.

4. $z = e^{2x} - 2x + 2y^2 + 3$

FOC:

$$f_x = 2e^{2x} - 2 = 0 \rightarrow e^{2x} = 1 \rightarrow 2x = \ln 1 = 0$$
$$f_y = 4y = 0$$

Critical point: $(0, 0)$

SOC:

$$f_{xx} = 4e^{2x} \rightarrow f_{xx}(0,0) = 4$$
$$f_{yy} = 4$$
$$f_{xy} = 0$$

At $(0, 0)$:

$$f_{xx} > 0, f_{yy} > 0$$
$$f_{xx}f_{yy} = 16 > 0 = f_{xy}^2 \rightarrow \text{ local minimum}$$

5. $z = (x-2)^4 + (y-3)^4$

(a) First note that $z \geqslant 0$ as square terms are always positive.

Since, $f(2,3) = 0$, z takes minimum value at $x^* = 2$ & $y^* = 3$.

(b) $f_x = 4(x-2)^3 \rightarrow f_x(2,3) = 0$

$f_y = 4(y-3)^3 \rightarrow f_y(2,3) = 0$

Yes, FOC is satisfied.

(c) $f_{xx}(2,3) = 12(x-2)^2 \rightarrow f_{xx}(2,3) = 0$

$f_{yy}(2,3) = 12(y-3)^2 \rightarrow f_{yy}(2,3) = 0$

$f_{xy} = 0$

SOC is not satisfied.

Yes, the second-order necessary condition is satisfied.