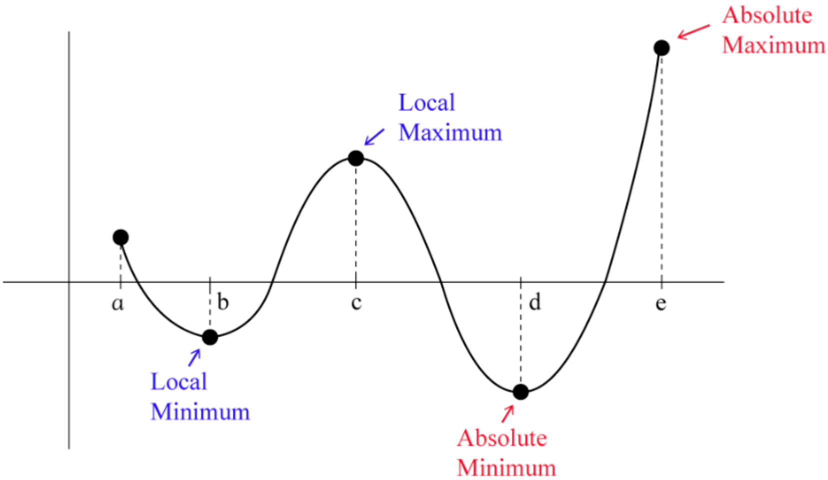
ECON 441

Introduction to Mathematical Economics

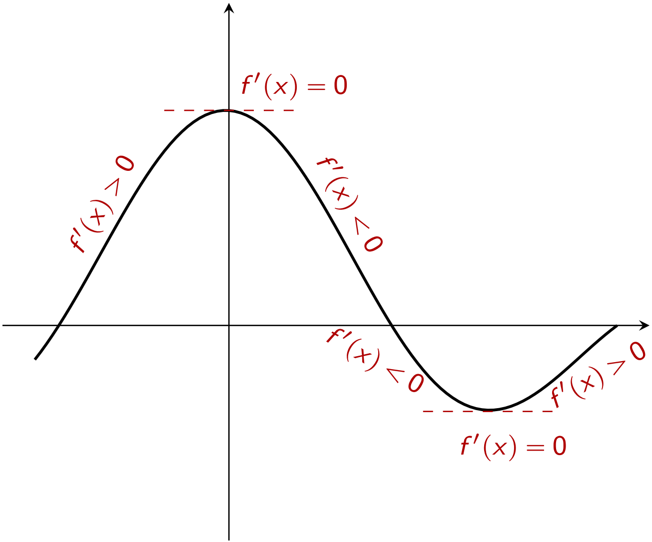
Div Bhagia

Lecture 9
Multivariable Optimization

Global vs Local Extrema



First-Derivative Test



Necessary vs Sufficient Conds.

Condition	Maximum	Minimum
First-order necessary	$f'(x) = 0$	$f'(x) = 0$
Second-order necessary †	$f''(x) \leqslant 0$	$f''(x) \geqslant 0$
Second-order sufficient †	$f''(x) < 0$	$f''(x) > 0$

† Applicable only after the first-order necessary condition has been satisfied.

Concave and Convex Functions

- Concave function: $f''(x) \leqslant 0$ for all x
- Convex function: $f''(x) \geqslant 0$ for all x

- Strictly concave function: $f''(x) < 0$ for all x
- Strictly convex function: $f''(x) > 0$ for all x

Global Optimizers

- If a function is concave, any critical point will give us a global maximum.
- If a function is strictly concave, any critical point will give us the *unique* global maximum.
- If a function is convex, any critical point will give us a global minimum.
- If a function is strictly convex, any critical point will give us the *unique* global minimum.

Example

$$y = 3x^2 + 3$$

Example

$$f : \mathbb{R}^+ \to \mathbb{R} \qquad f(x) = x^3 - 3x + 5$$

Example

$$f(x) = x + \frac{1}{x}$$

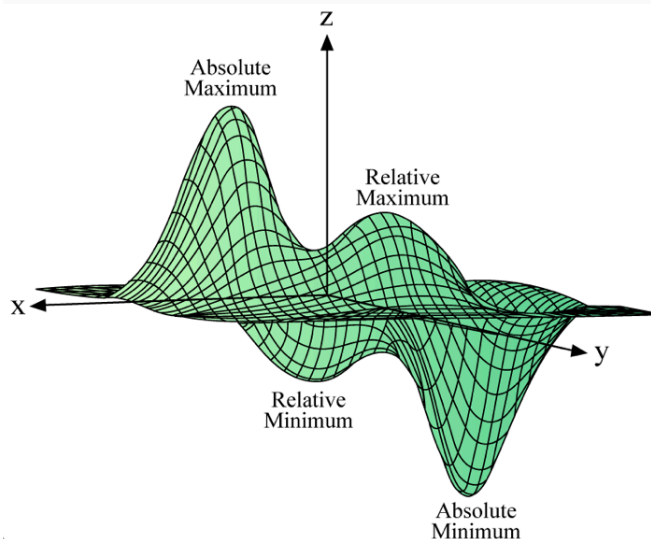
More than One Choice Variable

$$z = f(x, y)$$

What pair of values for x and y maximize/minimize the above function?

We will continue restricting ourselves to continuous functions that have continuous first-derivatives.

More than One Choice Variable



First-Order Conditions

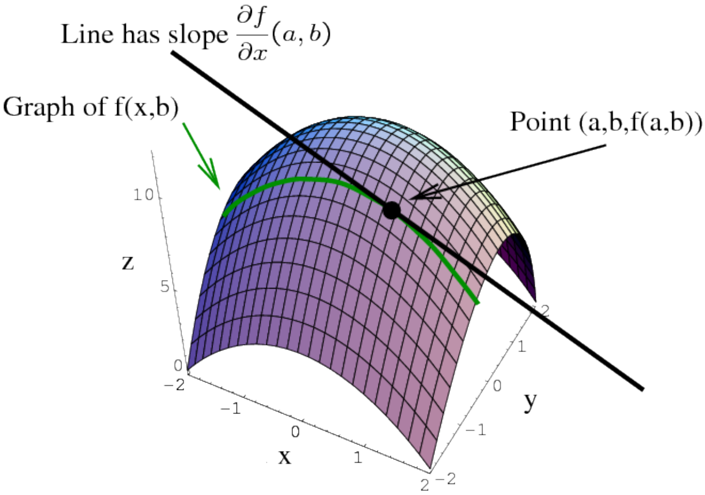
For the function

$$z = f(x, y)$$

The first order (necessary) condition:

$$f_x = f_y = 0$$

Partial Derivative

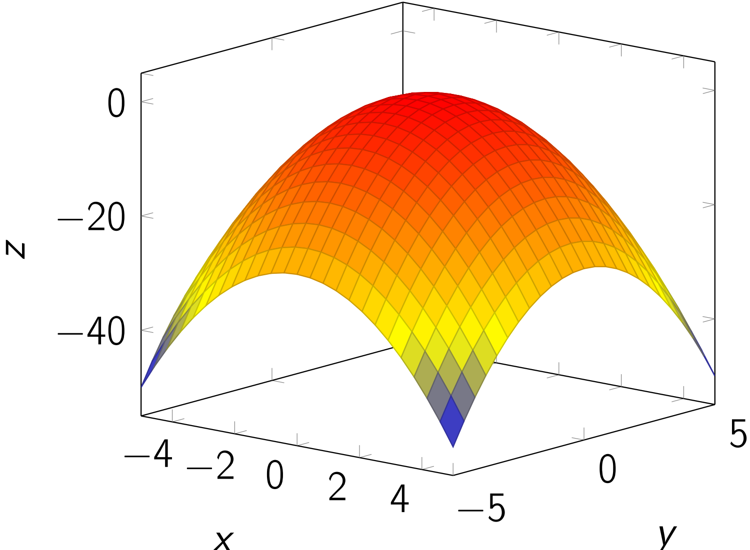


Example

What are the critical points for

$$f(x, y) = -(x^2 + y^2)$$

$$f(x, y) = -(x^2 + y^2)$$



Example

What are the critical points for

$$\pi(L, K) = f(K, L) - rK - wL$$

Second-Order Partial Derivatives

For the function

$$z = f(x, y)$$

$$f_{xx} \equiv \frac{\partial}{\partial x} f_x \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$f_{yy} \equiv \frac{\partial}{\partial y} f_y \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Second-Order Partial Derivatives

Also, have cross (or mixed) second-order partial derivatives.

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

We always have $f_{xy} = f_{yx}$ as long as f_{xy} and f_{yx} are both continuous.

Example

Find the four second-order partial derivatives of:

$$z = x^3 + 5xy - y^2$$

OLS

$$\min_{\{\alpha,\beta\}} \quad f(\alpha,\beta) = \sum_{i=1}^{n}(Y_i - \alpha - \beta X_i)^2$$

First-order conditions:

$$\frac{\partial f}{\partial \alpha} = -2\sum_{i=1}^{n}(Y_i - \alpha - \beta X_i) = 0$$

$$\frac{\partial f}{\partial \beta} = -2\sum_{i=1}^{n}(Y_i - \alpha - \beta X_i)X_i = 0$$

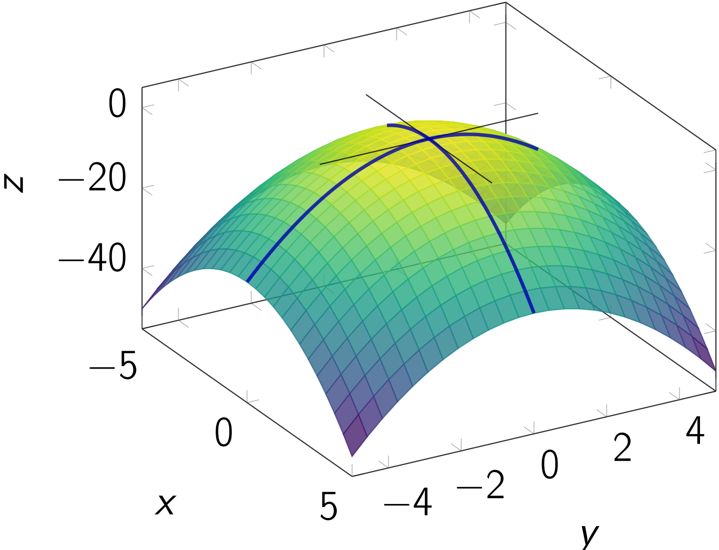
Second-Order Condition

Second-order (sufficient) conditions:

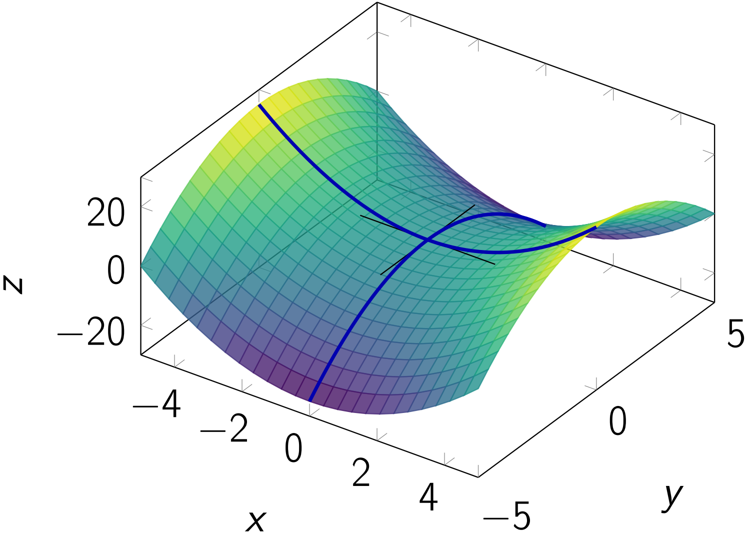
For maximum: $f_{xx} < 0, f_{yy} < 0,\ f_{xx}f_{yy} > \left(f_{xy}\right)^2.$

For minimum: $f_{xx} > 0, f_{yy} > 0,\ f_{xx}f_{yy} > \left(f_{xy}\right)^2.$

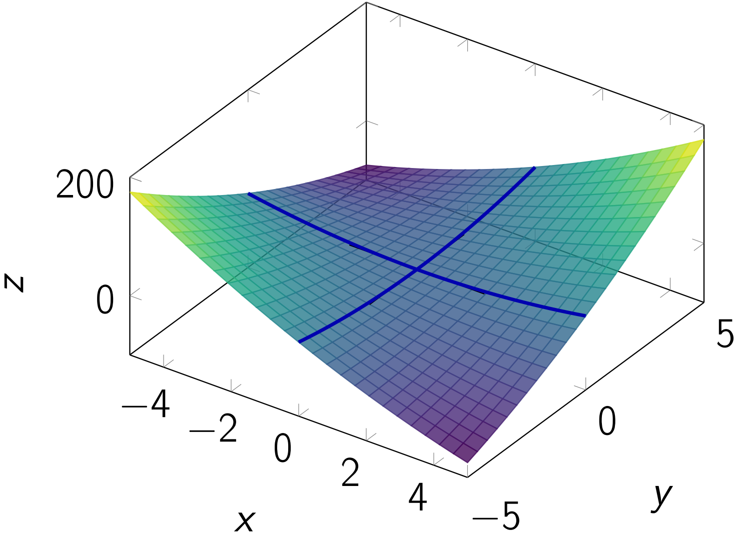
$$f(x,y) = -(x^2 + y^2)$$



$$f(x, y) = x^2 - y^2$$



$f(x, y) = x^2 + y^2 + 5xy$



Hessian Matrix

For the function:

$$y = f(x_1, x_2, ..., x_n)$$

The gradient vector ∇f and Hessian matrix H is given by

$$\nabla f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \qquad H = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{bmatrix}$$

More than Two Choice Variables

Condition	Maximum	Minimum
First-order necessary	$f_1 = f_2 = \ldots f_n = 0$ i.e. $\nabla f = 0$	$f_1 = f_2 = \ldots f_n = 0$ i.e. $\nabla f = 0$
Second-order sufficient	$\|H_1\| < 0, \|H_2\| > 0,$ $\|H_3\| < 0, \ldots$	$\|H_1\|, \|H_2\|, \ldots, \|H_n\| > 0$

† Applicable only after the first-order necessary condition has been satisfied.

References and Homework

- New sections today: Sections 11.1, 11.2
- Homework Problems: Exercise 11.2 1-5
- Reminder: Quiz 4 next week