Homework 4 Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Exercise 5.3

1.

$$\begin{vmatrix} 4 & 0 & -1 \\ 2 & 1 & -7 \\ 3 & 3 & 9 \end{vmatrix} = 4\begin{vmatrix} 1 & -7 \\ 3 & 9 \end{vmatrix} - 0\begin{vmatrix} 2 & -7 \\ 3 & 9 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix}$$

$$= 4(9+21) - 0(18+21) - 1(6-3)$$

$$= 120 - 0 - 3 = 117$$

Interchanging rows and columns:

$$\begin{vmatrix} 4 & 2 & 3 \\ 0 & 1 & 3 \\ -1 & -7 & 9 \end{vmatrix} = 4\begin{vmatrix} 1 & 3 \\ -7 & 9 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ -1 & 9 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ -1 & -7 \end{vmatrix}$$

$$= 4(9+21) - 2(0+3) + 3(0+1)$$

$$= 120 - 6 + 3 = 117$$

Interchange row 1 and 2:

$$\begin{vmatrix} 2 & 1 & -7 \\ 4 & 0 & -1 \\ 3 & 3 & 9 \end{vmatrix} = 2\begin{vmatrix} 0 & -1 \\ 3 & 9 \end{vmatrix} - 1\begin{vmatrix} 4 & -1 \\ 3 & 9 \end{vmatrix} - 7\begin{vmatrix} 4 & 0 \\ 3 & 3 \end{vmatrix}$$

$$= 2(0+3) - 1(36+3) - 7(12-0)$$

$$= 6 - 39 - 84 = -117$$

You can verify the other two properties in the same way.

5. (a)

$$
\begin{aligned}
&4\begin{vmatrix} 1 & -3 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 19 & 1 \\ 7 & 1 \end{vmatrix} \\
&= 4(0+3) + 1(19-7) \\
&= \quad 12 + 12 = 24
\end{aligned}
$$

Non-singular, rank = 3.

(b)

$$
\begin{aligned}
&5\begin{vmatrix} -2 & 1 \\ 0 & 3 \end{vmatrix} + 6\begin{vmatrix} 4 & 1 \\ 7 & 3 \end{vmatrix} \\
&=5(-6-0) + 6(12-7) \\
&= -30 + 30 = 0
\end{aligned}
$$

Singular. Rank<3, will need to put in echelon form to find exact rank.

(c)

$$
\begin{aligned}
&7\begin{vmatrix} 1 & 4 \\ -3 & -4 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 \\ 13 & -4 \end{vmatrix} \\
&=7(-4+12) + 1(-4-52) \\
&=56 - 56 = 0
\end{aligned}
$$

Singular. Rank <3, will need to put in echelon form to find exact rank.

(d)

$$
\begin{aligned}
&-3\begin{vmatrix} 9 & 5 \\ 8 & 6 \end{vmatrix} - 1\begin{vmatrix} -4 & 9 \\ 10 & 8 \end{vmatrix} \\
&= -3(54-40) - 1(-32-90) \\
&= -42 + 122 = 80
\end{aligned}
$$

Non-singular, rank = 3.

8. (a) False. While we can find the transpose of any matrix, the determinant is only defined for square matrices.

(b) False. Multiplying each element of an $n \times n$ by 2 will increase the determinant by 2^n times.

(c) A cannot be 0, but the determinant of A can. In any case, if $|A|$=0, A is singular.

Exercise 5.4

2. Note that

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

Here, $\text{adj } A = C'$ where $C = \left[|C_{ij}| \right]$.

(a)

$$|A| = 5 - 0 = 5$$

$$\text{adj } A = \begin{bmatrix} |C_{11}| & |C_{21}| \\ |C_{12}| & |C_{22}| \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 \\ 0 & 1 \end{bmatrix}$$

(b)

$$|B| = -2 - 0 = -2$$

$$B^{-1} = \frac{-1}{2} \begin{bmatrix} 2 & 0 \\ -9 & -1 \end{bmatrix}$$

(c)

$$|C| = -3 - 21 = -24$$

$$C^{-1} = \frac{-1}{24} \begin{bmatrix} -1 & -7 \\ -3 & 3 \end{bmatrix}$$

(d)

$$|D| = 21 - 0 = 21$$

$$D^{-1} = \frac{1}{21} \begin{bmatrix} 3 & -6 \\ 0 & 7 \end{bmatrix}$$

3. (a) Step 1: Exchange the two diagonal elements.
Step 2: Multiply both the off-diagonal elements by -1.

(b) Step 3: Multiply the resulting matrix from steps 1 and 2 by $1/|A|$.

4. (a)

$$|E| = 1\begin{vmatrix} 7 & 3 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 4 & -2 \\ 7 & 3 \end{vmatrix}$$
$$= 1(0-6) + 1(12+14)$$
$$= -6 + 26 = 20$$

$$\text{adj}\, E = \begin{bmatrix} |C_{11}| & |C_{21}| & |C_{31}| \\ |C_{12}| & |C_{22}| & |C_{32}| \\ |C_{13}| & |C_{23}| & |C_{33}| \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 2 & -3 \\ -7 & 2 & 7 \\ -6 & -4 & 26 \end{bmatrix}$$

$$E^{-1} = \frac{1}{|E|} adjE$$

(b)

$$F^{-1} = \frac{-1}{10}\begin{bmatrix} 0 & 2 & -3 \\ 10 & -6 & -1 \\ 0 & -4 & 1 \end{bmatrix}$$

(c)

$$|C_{11}| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \quad |C_{12}| = -\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \quad |C_{13}| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$|C_{21}| = -\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 \quad |C_{22}| = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \quad |C_{23}| = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$$

$$|C_{31}| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \quad |C_{32}| = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \quad |C_{33}| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$G^{-1} = -1\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(d)

$$H^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. (a)

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \quad v = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 28 \\ 42 \end{bmatrix}$$

$$\because Av = b \Rightarrow A^{-1}Av = A^{-1}b \Rightarrow v = A^{-1}b$$

$|A| = 20 - 6 = 14$

$A^{-1} = \frac{1}{14} \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$

$$\begin{aligned} v = A^{-1}b &= \frac{1}{14} \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 28 \\ 42 \end{bmatrix} \\ &= \frac{1}{14} \begin{bmatrix} 140 - 126 \\ -56 + 168 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 14 \\ 112 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \end{aligned}$$

So $x = 1$ and $y = 8$.

(b)

$$A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

$$\begin{aligned} |A| &= 4 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} - 1 \begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} - 5 \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} \\ &= 4(12 + 1) - 1(-8 - 3) - 5(2 - 9) \\ &= 52 + 11 + 35 = 98 \end{aligned}$$

$$\text{adj}\,A = \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}$$

$$x = A^{-1}b = \frac{1}{98}\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}_{3\times 3}\begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}_{3\times 1} = \frac{1}{98}\begin{bmatrix} 196 \\ 490 \\ 98 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

7. If a matrix is its own inverse, we would need that $A^2 = I$. This is true for the identity matrix. There are other possibilities such as matrix G in exercise 4.

Exercise 5.5

1 & 2. (a) $A = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \rightarrow |A| = 7 \quad b = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$

By Cramer's Rule:

$$A_1 = \begin{bmatrix} 6 & -2 \\ 11 & 1 \end{bmatrix} \rightarrow |A_1| = 28$$
$$A_2 = \begin{bmatrix} 3 & 6 \\ 2 & 11 \end{bmatrix} \rightarrow |A_2| = 21$$
$$x_1^* = \frac{|A_1|}{|A|} = \frac{28}{7} = 4$$
$$x_2^* = \frac{|A_2|}{|A|} = \frac{21}{7} = 3$$

Taking the inverse:

$$A^{-1} = \frac{1}{|A|}\text{adj} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$$
$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = A^{-1}b = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 28 \\ 21 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -1 \end{bmatrix} \rightarrow |A| = -11 \quad b = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$$

By Cramer's rule:

$$|A_1| = \begin{vmatrix} -3 & 3 \\ 12 & -1 \end{vmatrix} = -33 \rightarrow x_1^* = \frac{-33}{-11} = 3$$

$$|A_2| = \begin{vmatrix} -1 & -3 \\ 4 & 12 \end{vmatrix} = 0 \rightarrow x_2^* = \frac{0}{-11} = 0$$

By taking the inverse:

$$\begin{bmatrix} x_1^* \\ x_2 \end{bmatrix} = \frac{-1}{11} \begin{bmatrix} -1 & -3 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 12 \end{bmatrix} = \frac{-1}{11} \begin{bmatrix} -33 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

(c)

$$|A| = \begin{vmatrix} 8 & -7 \\ 1 & 1 \end{vmatrix} = 15$$

$$|A_1| = \begin{vmatrix} 9 & -7 \\ 3 & 1 \end{vmatrix} = 30 \quad \rightarrow x_1^* = \frac{30}{15} = 2$$

$$|A_2| = \begin{vmatrix} 8 & 9 \\ 1 & 3 \end{vmatrix} = 15 \rightarrow x_2^* = \frac{15}{15} = 1$$

Alternatively,

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 1 & 7 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(d)

$$|A| = \begin{vmatrix} 5 & 9 \\ 7 & -3 \end{vmatrix} = -78$$

$$|A_1| = \begin{vmatrix} 14 & 9 \\ 4 & -3 \end{vmatrix} = -78 \rightarrow x_1^* = 1$$

$$|A_2| = \begin{vmatrix} 5 & 14 \\ 7 & 4 \end{vmatrix} = -78 \rightarrow x_2^* = 1$$

Alternatively,

$$\begin{bmatrix} x_1^2 \\ x_2^* \end{bmatrix} = \frac{-1}{78}\begin{bmatrix} -3 & -9 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 14 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3. (a)

$$|A| = \begin{vmatrix} 8 & -1 & 0 \\ 0 & 2 & 5 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= 8\begin{vmatrix} 2 & 5 \\ 0 & 3 \end{vmatrix} - 1\begin{vmatrix} 0 & 5 \\ 2 & 3 \end{vmatrix} = 48 - 10 = 38$$

$$|A_1| = \begin{vmatrix} 16 & -1 & 0 \\ 5 & 2 & 5 \\ 7 & 0 & 3 \end{vmatrix}$$

$$16\begin{vmatrix} 2 & 5 \\ 0 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 5 \\ 7 & 3 \end{vmatrix} = 96 - 20 = 76$$

$$x_1^* = \frac{76}{38} = 2$$

$$|A_2| = \begin{vmatrix} 8 & 16 & 0 \\ 0 & 5 & 5 \\ 2 & 7 & 3 \end{vmatrix}$$

$$= 8\begin{vmatrix} 5 & 5 \\ 7 & 3 \end{vmatrix} - 16\begin{vmatrix} 0 & 5 \\ 2 & 3 \end{vmatrix}$$

$$= 8 \times -20 - 16 \times -10 = -160 + 160 = 0$$

$$x_2^* = 0$$

$$|A_3| = \begin{vmatrix} 8 & -1 & 16 \\ 0 & 2 & 5 \\ 2 & 0 & 7 \end{vmatrix}$$

$$= 8\begin{vmatrix} 2 & 5 \\ 0 & 7 \end{vmatrix} + 1\begin{vmatrix} 0 & 5 \\ 2 & 7 \end{vmatrix} + 16\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= 112 - 10 - 64 = 38 \rightarrow x_3^* = 1$$

(d)

$$|A| = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 0 + 2 + 2 = 4$$

$$|A_1| = \begin{vmatrix} a & 1 & 1 \\ b & -1 & 1 \\ c & 1 & -1 \end{vmatrix}$$

$$= a\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} b & 1 \\ c & -1 \end{vmatrix} + 1\begin{vmatrix} b & -1 \\ c & 1 \end{vmatrix}$$

$$= 0 - (-b - c) + (b + c) = 2(b + c)$$

$$|A_2| = \begin{vmatrix} -1 & a & 1 \\ 1 & b & 1 \\ 1 & c & -1 \end{vmatrix}$$

$$= -1\begin{vmatrix} b & 1 \\ c & -1 \end{vmatrix} - a\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix}$$

$$= -1(-b - c) - a(-1 - 1) + 1(c - b)$$

$$= b + c - 2a + c - b = 2(a + c)$$

$$|A_3| = \begin{vmatrix} -1 & 1 & a \\ 1 & -1 & b \\ 1 & 1 & c \end{vmatrix}$$

$$= -1\begin{vmatrix} -1 & b \\ 1 & c \end{vmatrix} - 1\begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix} + a\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= -1(-c - b) - 1(c - b) + 2a$$

$$= c + b - c + b + 2a = 2(a + b)$$

$$x^* = \frac{b+c}{2} \quad y^* = \frac{a+c}{2} \quad z^* = \frac{a+b}{2}$$