# ECON 441

## Introduction to Mathematical Economics

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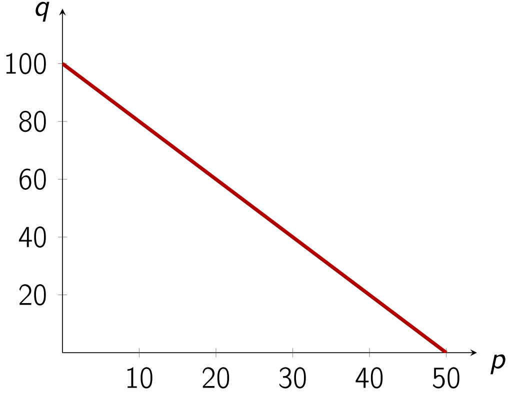
Lecture 2: Linear Algebra

# A Simple Economic Model

$q$: quantity of hats, $p$: price of a single hat

Demand for hats: $q = 100 - 2p$



# A Simple Economic Model

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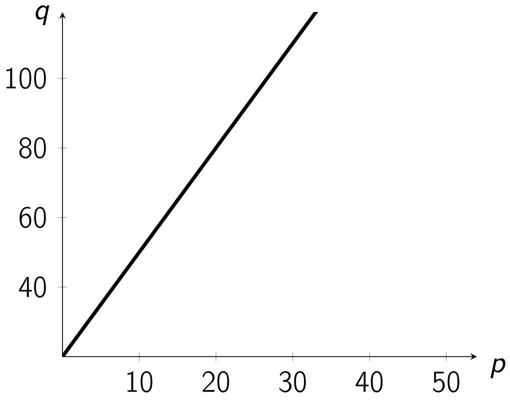
Supply for hats: $q = 20 + 3p$



# A Simple Economic Model

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*Equilibrium*: At what price will both demand and supply be equal?

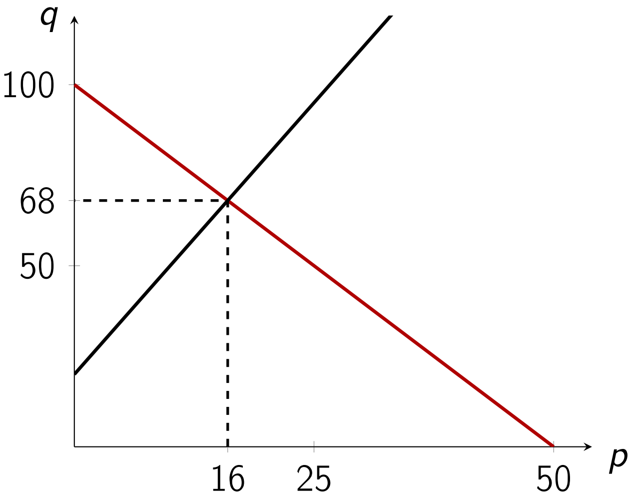
$$100 - 2p = 20 + 3p \rightarrow p^* = \$16$$

What is the quantity traded at this price?

$$q^* = 100 - 2 \times 16 = 20 + 3 \times 16 = 68$$

$q^*$ and $p^*$ are determined simultaneously.

# Equilibrium



# Matrix Algebra

We solved a *system* of two (linear) equations in two variables.

Complex economic models: multiple equations with multiple variables

Hard to just wing it... Enter, Matrix Algebra!

Matrix Algebra can help us write complex system of equations compactly and solve them

# A Simple Economic Model

$q$: quantity of hats, $p$: price of a single hat

Demand for hats: $q = 100 - 2p$

Supply for hats: $q = 20 + 3p$

Rewrite the two equations:

$$\begin{aligned} q + 2p &= 100 \\ q - 3p &= 20 \end{aligned}$$

# A Simple Economic Model

Two equations in two unknowns:

$$q + 2p = 100$$
$$q - 3p = 20$$

Can write this as:

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

These arrays are called matrices.

# Today

- Matrices: Addition, Subtraction, and Scalar Multiplication
- Matrix Multiplication
- Vectors
- Identity and Null Matrices
- Transpose and Inverse of a Matrix

Textbook reference: 4.1-4.6

# Matrices

A *matrix* is a rectangular array of numbers, parameters, or vectors.

Example. $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 6 \end{bmatrix}$

Dimensions of matrix:

- Number of rows ($m$)
- Number of columns ($n$)

# Matrices

A matrix with $m$ rows and $n$ columns is referred to as an $m \times n$ matrix

What's the dimension of $A$?

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 6 \end{bmatrix}$$

# Matrices

A matrix with $m$ rows and $n$ columns is referred to as an $m \times n$ matrix

What's the dimension of $A$?

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 6 \end{bmatrix}_{2\times 3}$$

# Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Can write it more compactly

$$A = [a_{ij}] \quad i = 1, 2, ..., m; j = 1, 2, ..., n$$

# Matrices

*Square matrix*: equal number of rows and columns

Example.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

# Matrices

Two matrices are *equal* if all their elements are identical.

Example.

$$A = \begin{bmatrix} 1 & 8 \\ 4 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & 8 \\ 4 & 2 \end{bmatrix}$$

So $A = B$ if and only if $a_{ij} = b_{ij}$ for all $i, j$

# Matrix Addition and Subtraction

- How to add or take the difference between two matrices?

  $\rightarrow$ Element-by-element

  $\rightarrow$ Matrices have to have same dimension

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}$$

- What is $A + B$ and $A - B$?

# Scalar Multiplication

How to multiply a scalar to a matrix?

$$\lambda \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix}$$

Example.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}$$

What is $2B$ and $A - 2B$?

# Matrix Multiplication

A whole new animal...

Only possible to multiply two matrices, $A_{m\times n}$ and $B_{p\times q}$ to get $AB$ if $n = p$ i.e.

$$\text{number of columns in } A = \text{ number of rows in } B$$

Example. $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} 1 & 8 \\ -2 & 3 \end{bmatrix}_{2\times 2}$

Cannot do $AB$, but can do $BA$

# Matrix Multiplication

Another example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}_{3\times 1}$$

Can we multiply $A$ and $B$ to find $C = AB$?

# Matrix Multiplication

Another example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}_{3\times 1}$$

Can we multiply $A$ and $B$ to find $C = AB$?

Yes, since $A$ has 3 rows which is equal to the number of columns in $B$.

Also, the dimension of $C$ will be $2 \times 1$.

# Matrix Multiplication

So how to actually multiply these matrices?

$$C = AB$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

The element $c_{ij}$ is obtained by multiplying term-by-term the entries of the $i$th row of $A$ and $j$th column of $B$.

# Matrix Multiplication: Examples

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$$

Here,

$$C = AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}_{2\times 2}$$

# Matrix Multiplication: Examples

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}_{3\times 1}$$

# Matrix Multiplication: Examples

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & -2 \end{bmatrix}_{2\times 3} \quad B = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}_{3\times 1}$$

$$C = AB = \begin{bmatrix} 2\times 1 + 3\times -2 + 1\times 4 \\ 4\times 1 + -6\times -2 + -2\times 4 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}_{2\times 1}$$

# Matrix Multiplication: Examples

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

# A Simple Economic Model

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

What is $Ax$?

# A Simple Economic Model

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

What is $Ax$?

$$Ax = \begin{bmatrix} q + 2p \\ q - 3p \end{bmatrix}$$

# A Simple Economic Model

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

What is $Ax$?

$$Ax = \begin{bmatrix} q + 2p \\ q - 3p \end{bmatrix}$$

Setting $Ax = b$ gives us back our demand and supply equations.


HEY A.I. WHY DO YOU ALWAYS WEAR THAT MASK?
A.I.
A.I.
A.I.
$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$
LET'S KEEP THIS ON.
A.I.
OWLTURD.COM

# Vectors

- Matrices with only one column: column vectors

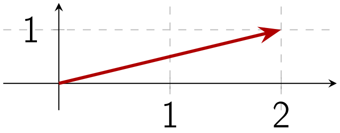
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Matrices with only one row: row vectors

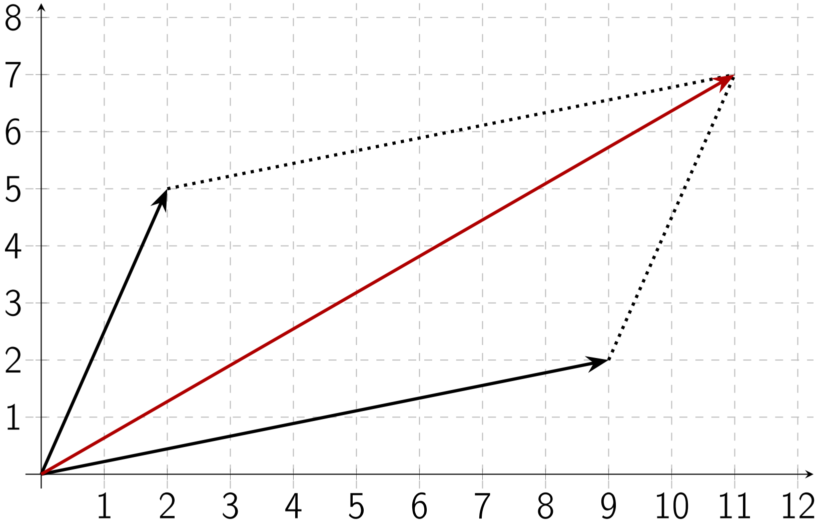
$$x' = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$$

# Geometric Representation

- Vectors can be graphically represented by arrows
- The direction of the arrow indicates the vector's *direction* and the length of the arrow corresponds to its *magnitude*.
- Here is a plot for vector $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



# Addition of Vectors



# Inner Product

Inner product of two vectors each with $n$ elements:

$$u \cdot v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

Example.

$$u = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \qquad v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

# Linear Dependence

A set of vectors is said to be *linearly dependent* if and only if any one of them can be expressed as a linear combination of the remaining vectors.

Example.

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

# Linear Dependence

A set of vectors is said to be *linearly dependent* if and only if any one of them can be expressed as a linear combination of the remaining vectors.

Example.

$$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

# Linear Dependence

A set of $m$-vectors $v_1, v_2, ..., v_n$ is linearly dependent if and only if there exists a set of scaler $k_1, k_2, ..., k_n$ (not all zero) such that:

$$\sum_{i=1}^{n} k_i v_i = 0 \quad (m \times 1)$$

# Identity Matrices

Square matrix with $1$s in its *principal diagonal* and $0$s elsewhere

A $2 \times 2$ identity matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A $3 \times 3$ identity matrix:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Identity Matrices

Acts like 1,

$$AI = IA = A$$

Example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & 2 \end{bmatrix}$$

# Idempotent Matrices

A matrix is an *idempotent* matrix if it remains unchanged when multiplied by itself any number of times.

$A$ is idempotent if and only if $A = A^k$.

Is an identity matrix idempotent?

# Null Matrix

A null matrix is a matrix with all elements $0$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $A + 0 = A$
- $A0 = 0$

# Transpose of a Matrix

Transpose of A ($A'$ or $A^T$): interchange rows and columns

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -6 & 2 \end{bmatrix}$$

# Transpose of a Matrix

- A matrix $A$ is said to be *symmetric* if

$$A' = A$$

- A matrix $A$ is said to be *skew-symmetric* if

$$A' = -A$$

- A matrix $A$ is said to be *orthogonal* if

$$A'A = I$$

# Example: Symmetric Matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -5 \\ 0 & -5 & 4 \end{bmatrix}$$

# Example: Skew-symmetric Matrix

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

# Example: Orthogonal Matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

# Properties of Transposes

$$(A')' = A$$

$$(A + B)' = A' + B'$$

$$(AB)' = B'A'$$

Example: $A = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$

# Inverse of a Matrix

For a **square** matrix $A$, it's inverse $A^{-1}$ is defined as:

$$AA^{-1} = A^{-1}A = I$$

Squareness is a *necessary* condition not a *sufficient* condition

If a matrix's inverse exists, it's called a **nonsingular** matrix

# Properties of Inverses

$$\left(A^{-1}\right)^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A')^{-1} = \left(A^{-1}\right)'$$

# Solution of Linear-Equation System

$$Ax = b$$

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$$Ax = b$$

Pre-multiply both sides by $A^{-1}$,

$$A^{-1}Ax = A^{-1}b \quad \implies x = A^{-1}b$$

# Solution of Linear-Equation System

$$Ax = b$$

Pre-multiply both sides by $A^{-1}$,

$$A^{-1}Ax = A^{-1}b \quad \Longrightarrow x = A^{-1}b$$

If $A$ is singular, a unique solution does not exist.

# Conditions for Nonsingularity

Squareness is *necessary* but not *sufficient*

Sufficient condition for nonsingularity:

> *Rows (or equivalently) columns are linearly independent*

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

# Conditions for Nonsingularity

Squareness is *necessary* but not *sufficient*

Sufficient condition for nonsingularity:

> *Rows (or equivalently) columns are linearly independent*

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$A$ is singular, $B$ is nonsingular.

# Conditions for Nonsingularity

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad d = \begin{bmatrix} a \\ b \end{bmatrix}$$

We have a system of linear equations:

$$Ax = d$$

Then,

$$x_1 + 2x_2 = a$$
$$2x_1 + 4x_2 = b$$

# Conditions for Nonsingularity

$$x_1 + 2x_2 = a$$
$$2x_1 + 4x_2 = b$$

For these equations to be consistent, we need $b = 2a$:

$$x_1 + 2x_2 = a$$

$$2x_1 + 4x_2 = 2a$$

Both are the same equation, infinite number of solutions.

# Conditions for Nonsingularity

To summarize, for a matrix to be nonsingular (i.e. its inverse exists):

Necessary condition: **Squareness**

Sufficient condition: **Rows or (equivalently) columns are linearly independent**

# Homework Problems

- Exercise 4.2: 1, 2, 4
- Exercise 4.4: 5 (e), 7
- Exercise 4.5: 1, 4
- Exercise 4.6: 2, 6

Reminder: Quiz 1 is next week.