ECON 441

Introduction to Mathematical Economics

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Lecture 10
Constrained Optimization

Utility Maximization

Maximize:

$$f(x_1, x_2) = x_1 x_2$$

Subject to the budget constraint:

$$2x_1 + x_2 = 40$$

To find x_1^* and x_2^*, we could substitute $x_2 = 40 - 2x_1$ and use our first-order and second-order conditions.

Utility Maximization

$$f(x_1, x_2) = x_1(40 - 2x_1) = 40x_1 - 2x_1^2$$

Constrained Maximization

Substitution doesn't always work, get's complicated with many variables and constraints.

Thankfully, we have the method of Lagrange multiplier.

Lagrange function:

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(40 - 2x_1 - x_2)$$

λ is the Lagrange multiplier.

Now apply FOC for the case of 3 variables (x_1, x_2, λ).

Utility Maximization

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(40 - 2x_1 - x_2)$$

First-order conditions:

$$\frac{\partial L}{\partial x_1} = x_2 - 2\lambda = 0$$
$$\frac{\partial L}{\partial x_2} = x_1 - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 40 - 2x_1 - x_2 = 0$$

Lagrange-Multiplier Method

Given an objective function

$$y = f(x_1, x_2)$$

subject to the constraint

$$g(x_1, x_2) = c$$

where c is a constant. We can write the Lagrange function as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda[c - g(x_1, x_2)]$$

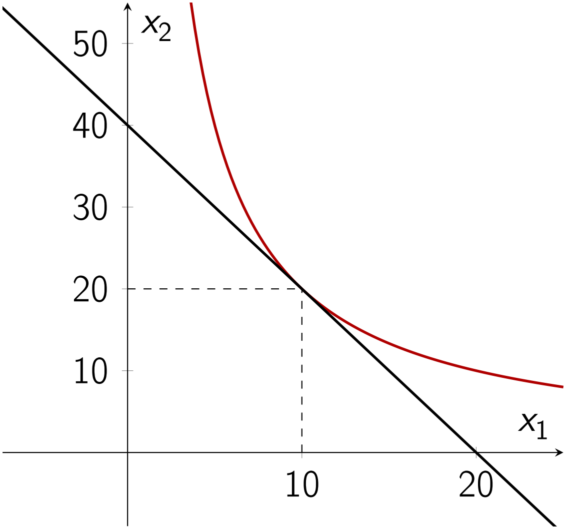
Lagrange-Multiplier Method

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda[c - g(x_1, x_2)]$$

First-order conditions:

$$\frac{\partial L}{\partial x_1} = f_1(x_1^*, x_2^*) - \lambda^* \cdot g_1(x_1^*, x_2^*) = 0$$
$$\frac{\partial L}{\partial x_2} = f_2(x_1^*, x_2^*) - \lambda^* \cdot g_2(x_1^*, x_2^*) = 0$$
$$\frac{\partial L}{\partial \lambda} = c - g(x_1^*, x_2^*) = 0$$

Utility Maximization



At the optimal point:

$$\frac{\partial f/\partial x_1}{\partial f/\partial x_2} = \frac{\partial g/\partial x_1}{\partial g/\partial x_2}$$

Balance trade-off:
optimize f vs satisfy g

Interpretation of the Lagrange Multiplier

At the optimum:

$$L(x_1^*, x_2^*, \lambda^*) = f(x_1^*, x_2^*) + \lambda^*[c - g(x_1^*, x_2^*)]$$

What if c changes by a little bit?

Interpretation of the Lagrange Multiplier

At the optimum:

$$L(x_1^*, x_2^*, \lambda^*) = f(x_1^*, x_2^*) + \lambda^*[c - g(x_1^*, x_2^*)]$$

What if c changes by a little bit?

$$\begin{aligned}\frac{dL^*}{dc} &= f_1 \cdot \frac{dx_1^*}{dc} + f_2 \cdot \frac{dx_2^*}{dc} + \lambda^* \left[1 - g_1 \cdot \frac{dx_1^*}{dc} - g_1 \cdot \frac{dx_1^*}{dc}\right] \\ &= \underbrace{(f_1 - \lambda^* \cdot g_1)}_{0} \cdot \frac{dx_1^*}{dc} + \underbrace{(f_2 - \lambda^* \cdot g_2)}_{0} \cdot \frac{dx_2^*}{dc} + \lambda^* = \lambda^*\end{aligned}$$

Utility Maximization

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(40 - 2x_1 - x_2)$$

At the optimum:

$$x_2* = 2x_1^*, \quad 2x_1^* + x_2^* = 40 \quad \rightarrow \quad x_1^* = 10, x_2^* = 20, \lambda^* = 10$$

If my income increases by $1, by how much does my utility increase? $\lambda^* = 10$

Example I

$$\max_{\{x_1, x_2\}} \quad x_1^2 x_2 \quad s.t. \quad 2x_1^2 + x_2^2 = 3$$

Example II

$$\max_{\{x,y\}} \quad x^a y^b \quad s.t. \quad x + y = 10$$

Example III

$$\max_{\{x_1, x_2\}} \quad x_1^{1/3} x_2^{2/3} \quad s.t. \quad x_1 + 4x_2 = 30$$

More Generally

$$\max_{\{x_1, x_2\}} \quad x_1^{\alpha} x_2^{\beta} \quad s.t. \quad p_1 x_1 + p_2 x_2 = m$$

Multiple Variables

Objective function:

$$\text{Maximize } f(x_1, x_2, \cdots, x_n) \quad \text{subject to} \quad g(x_1, x_2, \cdots, x_n) = c$$

Lagrange function:

$$L = f(x_1, x_2, \cdots, x_n) + \lambda [c - g(x_1, x_2, \cdots, x_n)]$$

First-order conditions:

$$L_i = f_i(x_1, x_2, \cdots, x_n) - \lambda g_i(x_1, x_2, \cdots, x_n) \quad [i = 1, 2, \cdots, n]$$
$$L_\lambda = c - g(x_1, x_2, \cdots, x_n)$$

Multiple Constraints

Maximize $f(x_1, x_2, \cdots, x_n)$, subject to

$$g(x_1, x_2, \cdots, x_n) = c \quad \text{and} \quad h(x_1, x_2, \cdots, x_n) = d$$

Lagrange function:

$$L = f(x_1, x_2, \cdots, x_n) + \lambda [c - g(x_1, x_2, \cdots, x_n)] + \mu [d - h(x_1, x_2, \cdots, x_n)]$$

First-order conditions:

$$L_\lambda = c - g(x_1, x_2, \cdots, x_n) = 0$$
$$L_\mu = d - h(x_1, x_2, \cdots, x_n) = 0$$
$$L_i = f_i(x_1, x_2, \cdots, x_n) - \lambda g_i(x_1, x_2, \cdots, x_n) - \mu h_i(x_1, x_2, \cdots, x_n) = 0$$

Example IV

$$\max_{\{x,y,z\}} \quad yz + xz \quad s.t. \quad y^2 + z^2 = 1 \quad and \quad xz = 3$$

Intertemporal Utility Maximization

$$U = U(c_1, c_2) = \ln c_1 + \beta \ln c_2 \qquad 0 < \beta < 1$$

- $y_1, y_2 > 0$: income in period 1 and 2
- Income you save s in period 1 earns interest $r > 0$
- In which case,

$$c_1 + s = y_1 \qquad c_2 = y_2 + (1 + r)s$$

- Combining these constraints:

$$c_1 + \frac{1}{1+r} c_2 = \underbrace{y_1 + \frac{1}{1+r} y_2}_{m \equiv \text{present-discounted income}}$$

Intertemporal Utility Maximization

$$\max_{\{c_1, c_2\}} \quad U(c_1, c_2) = \ln c_1 + \beta \ln c_2 \quad s.t. \quad c_1 + \frac{1}{1+r} c_2 = m$$

Another Example: Lasso Regression

- Regularization is a fundamental concept in machine learning and statistics, used to improve the generalization of models to new data by preventing overfitting
- Overfitting occurs when a model learns the detail and noise in the training data
- Regularization techniques modify the learning algorithm to impose constraints on the model parameters, typically by adding a penalty term to the loss function

Lasso Regression

Linear regression, unconstrained maximization:

$$L(\beta) = \sum_{i=1}^{n} (Y_i - X_i^T \beta)^2$$

Lasso regression, constrained maximization:

$$L(\beta) = \sum_{i=1}^{n} (Y_i - X_i^T \beta)^2 + \lambda \sum_{j=1}^{p} |\beta_j|$$

Higher values of λ increase the penalty for large coefficients.

References and Homework Problems

- References for today: Sections 12.1 and 12.2
- Homework: Exercise 12.2: Questions 1–4 + Examples from today's class