Sample Final Exam

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Print Name: ____________________

This is a closed-book test. You may not use a phone or a computer.

Time allotted: 110 minutes
Total points: 40

Please show sufficient work so that the instructor can follow your work.

I understand and will uphold the ideals of academic honesty as stated in the honor code.

Signature: ____________________

1. (6 pts) Answer the following questions. (1 pt each)

   (a) What is the inverse of the function $f(x) = 4x + 6$?

   (b) Find the intersection of the following sets:

   $$A = \{x : x > 0\} \qquad B = \{x : x \text{ is an even number}\}$$

   (c) The inverse of a 4×4 matrix A exists if

   - ☐ The determinant of A is 0
   - ☐ The determinant of A is not 0
   - ☐ Rank of A is 4
   - ☐ Rank of A is 0
   - ☐ All rows of A are linearly independent

   Select all that apply.

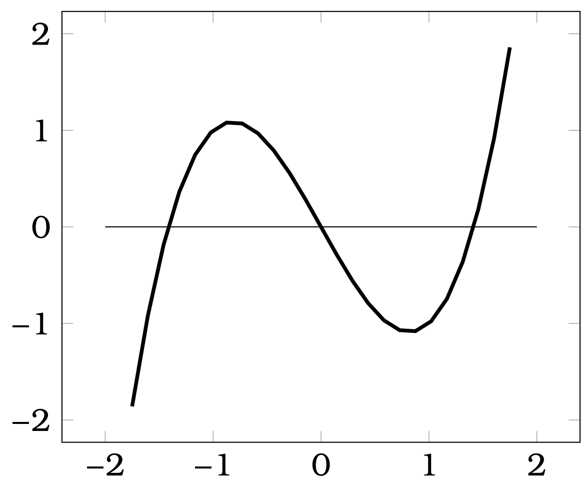
(d) Is the function $y = |x - 1|$ differentiable at $x = 1$?

- ☐ Yes.
- ☐ No.
- ☐ Can't say.

(e) The function $f(K, L) = K^{\alpha} L^{1-\alpha}$ is:

- ☐ Homogeneous of degree 0
- ☐ Homogeneous of degree 1
- ☐ Not homogeneous
- ☐ Homogeneous but cannot say of what degree

(f) The following function is:



- ☐ Quasiconcave
- ☐ Strictly quasiconcave
- ☐ Quasiconvex
- ☐ None of the above

2. (5 pts) Consider a single-variable function:

$$y = f(x)$$

In class, we learnt that at any local maximum or minimum, we must have that,

$$f'(x^*) = 0$$

In addition, a sufficient condition for a critical point x^* to be a local maximizer is:

$$f''(x^*) < 0$$

(a) (2 pts) Why can't we have a maximum or minimum at a point where $f'(x) > 0$ or $f'(x) < 0$?

(b) (2 pts) Why is $f''(x) < 0$ a sufficient condition for a critical point to be a maximizer?

(c) (1 pt) If f is a strictly concave function, can we have two critical points, i.e., two distinct values of x such that $f'(x_1) = f'(x_2) = 0$?

3. (5 pts) Prove the following statements:

   (a) (2 pts) $\sum_{i=1}^{2} 3(x_i + 1) = 3\left(\sum_{i=1}^{2} x_i\right) + 6$

   (b) (3 pts) Given the following production function:

$$Q = f(K, L) = AK^{\alpha}L^{\beta}$$

   The partial elasticity of output with respect to capital K and labor L is α and β, respectively.

4. (8 pts) *Consider the following system of equations:*

$$x_1 + 2x_2 = 6$$
$$3x_1 + x_2 = 3$$

   (a) (2 pts) *Write this system of equations in matrix format i.e.,*

$$Av = b$$

   What is A, v, and b equal to?

   (b) (2 pts) *Calculate the inverse of A.*

(c) (2 pts) *If you premultiply A^{-1} on both sides of the equation $Av = b$, you should be able to derive an expression to solve for v. Write down this expression.*

(d) (2 pts) *Using the expression in (c) solve for v^*.*

5. (16 pts) You are given the following inter-temporal utility function:

$$U = U(c_1, c_2) = \ln c_1 + \beta \ln c_2$$

where c_1 and c_2 is consumption in period 1 and 2, respectively. $0 < \beta < 1$ is the rate at which you discount the future and it measures your impatient. You earn income $y_1 > 0$ in period 1 and income $y_2 > 0$ in period 2. Any of the income you save s in period 1 earns interest $r > 0$. So,

$$c_1 + s = y_1, \qquad c_2 = y_2 + (1 + r)s$$

Combining these constraints (plugging $s = y_1 - c_1$ in the expression for c_2):

$$c_1 + \frac{1}{1+r} c_2 = y_1 + \frac{1}{1+r} y_2$$

Let the present-discounted income be denoted by m, such that:

$$m = y_1 + \frac{1}{1+r} y_2$$

You want to choose c_1 and c_2 to maximize utility

$$U = U(c_1, c_2) = \ln c_1 + \beta \ln c_2$$

subject to the constraint:

$$c_1 + \frac{1}{1+r} c_2 = m$$

(a) (2 pts) Write down the Lagrangian function corresponding to the above maximization problem.

(b) (3 pts) Write down the first-order conditions for a critical point.

(c) (3 pts) Using the first order conditions in (b), show that the optimal consumption c_1^* and c_2^* and the Lagrange multiplier λ^* are given by:

$$c_1^* = \frac{m}{1+\beta}, \quad c_2^* = \frac{\beta m(1+r)}{1+\beta}, \quad \lambda^* = \frac{1+\beta}{m}$$

(d) (1 pt) Here, $U(c_1, c_2)$ is a strictly quasiconcave function. Is this sufficient to conclude that the c_1^* and c_2^* we found in (c) characterize a global maximum?

(e) (2 pts) How does the optimal consumption in period 1 and 2 change due to an increase in m? Calculate $\partial c_1^*/\partial m$ and $\partial c_2^*/\partial m$ to answer your question.

(f) (1 pt) If $r = 0$, using your expressions for $\partial c_1^*/\partial m$ and $\partial c_2^*/\partial m$, answer whether optimal consumption in period 1 changes by more or less than consumption in period 2 in response to a change in m?

(g) (2 pts) How does optimal consumption in period 1 change due to an increase in the interest rate r?

Note that here,

$$m = y_1 + \frac{y_2}{1+r}$$

So to calculate $\partial c_1^* / \partial r$, you need to use the chain-rule as follows:

$$\frac{\partial c_1^*}{\partial r} = \frac{\partial c_1^*}{\partial m} \cdot \frac{\partial m}{\partial r}$$

(h) (2 pts) By how much does the maximum utility $U(c_1^*, c_2^*)$ change due to an increase in m? (Hint: What does λ^* tell us?)