Sample Final Exam Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Print Name: ____

This is a closed-book test. You may not use a phone or a computer.

Time allotted: 110 minutes
Total points: 40

Please show sufficient work so that the instructor can follow your work.

I understand and will uphold the ideals of academic honesty as stated in the honor code.

Signature: ____________________________

1. (5 pts) Answer the following questions. (1 pt each)

   (a) What is the inverse of the function $f(x) = 4x + 6$?

   $$f^{-1}(x) = g(x) = \frac{x-6}{4}$$

   (b) Find the intersection of the following sets:

   $$A = \{x : x > 0\} \qquad B = \{x : x \text{ is an even number}\}$$

   $$A \cap B = \{2, 4, 6, \ldots\}$$

   (c) The inverse of a 4×4 matrix A exists if

   - ☐ The determinant of A is 0
   - ☑ The determinant of A is not 0
   - ☑ Rank of A is 4
   - ☐ Rank of A is 0
   - ☑ All rows of A are linearly independent

   Select all that apply.

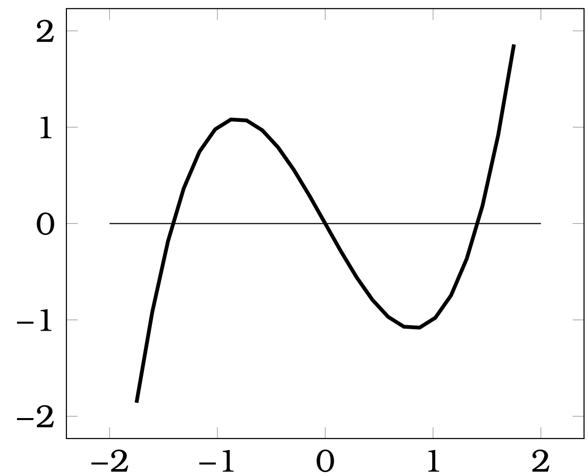
   (d) Is the function $y = |x - 1|$ differentiable at $x = 1$?

   - ☐ Yes.
   - ☑ No.
   - ☐ Can't say.

   (e) The function $f(K, L) = K^{\alpha} L^{1-\alpha}$ is:

   - ☐ Homogeneous of degree 0
   - ☑ Homogeneous of degree 1
   - ☐ Not homogeneous
   - ☐ Homogeneous but cannot say of what degree

(f) The following function is:



- ☐ Quasiconcave
- ☐ Strictly quasiconcave
- ☐ Quasiconvex
- ☑ None of the above

2. (5 pts) Consider a single-variable function:

$$y = f(x)$$

In class, we learnt that at any local maximum or minimum, we must have that,

$$f'(x^*) = 0$$

In addition, a sufficient condition for a critical point x^* to be a local maximizer is:

$$f''(x^*) < 0$$

(a) (2 pts) Why can't we have a maximum or minimum at a point where $f'(x) > 0$ or $f'(x) < 0$?

A point where $f'(x) > 0$ or $f'(x) < 0$ cannot be a local maximum or minimum because the value of the function will change in the neighborhood of such a point. For example, if $f'(x^*) > 0$, then increasing x slightly above x^* will lead to an increase in the value of the function, so x^* cannot be a local maximum. Similarly, decreasing x slightly below x^* will lead to a decline in the value of the function, so x^* cannot be a local minimum as well.

(b) (2 pts) Why is $f''(x) < 0$ a sufficient condition for a critical point to be a maximizer?

If x^* is a critical point, that is $f'(x^*) = 0$, and $f''(x^*) < 0$, we can conclude that x^* is a local maximizer. This is because $f''(x^*) < 0$ implies that $f'(x)$ is decreasing around x^*, and since $f'(x^*) = 0$, the sign of $f'(x)$ flips from positive to negative as we move from left to right of x^*. Since $f'(x) > 0$ indicates that $f(x)$ is increasing and $f'(x) < 0$ indicates that $f(x)$ is decreasing, it follows that $f(x)$ is increasing as we approach x^* and decreasing as we move away from it. Therefore, x^* corresponds to a peak in the function.

(c) (1 pt) If f is a strictly concave function, can we have two critical points, i.e., two distinct values of x such that $f'(x_1) = f'(x_2) = 0$?

No, if f is a strictly concave function, it cannot have two distinct critical points. This is due to the fact that $f''(x) < 0$ for all x in the domain of f, implying that $f'(x)$ is strictly decreasing. As a result, $f'(x)$ can only be zero at one point.

3. (5 pts) Prove the following statements:

   (a) (2 pts) $\sum_{i=1}^{2} 3(x_i + 1) = 3\sum_{i=1}^{2} x_i + 6$

$$\begin{aligned}\sum_{i=1}^{2} 3(x_i + 1) &= 3(x_1 + 1) + 3(x_2 + 1) \\ &= 3x_1 + 3 + 3x_2 + 3 \\ &= 3(x_1 + x_2) + 6 \\ &= 3\sum_{i=1}^{2} x_i + 6\end{aligned}$$

   (b) (3 pts) Given the following production function:

$$Q = f(K, L) = AK^{\alpha}L^{\beta}$$

   The partial elasticity of output with respect to capital K and labor L is α and β, respectively.

   Elasticity of output w.r.t. capital:

$$\varepsilon_{QK} = \frac{\partial Q}{\partial K} \cdot \frac{K}{Q} = \alpha AK^{\alpha-1}L^{\beta} \cdot \frac{K}{AK^{\alpha}L^{\beta}} = \alpha$$

   Elasticity of output w.r.t. labor:

$$\varepsilon_{QL} = \frac{\partial Q}{\partial L} \cdot \frac{L}{Q} = \beta AK^{\alpha}L^{\beta-1} \cdot \frac{L}{AK^{\alpha}L^{\beta}} = \beta$$

4. (6 pts) *Consider the following system of equations:*

$$x_1 + 2x_2 = 6$$
$$3x_1 + x_2 = 3$$

(a) (1 pt) *Write this system of equations in matrix format i.e.,*

$$Av = b$$

What is A, v, and b equal to?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

(b) (2 pts) *Calculate the inverse of A.*

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

(c) (1 pts) *If you premultiply A^{-1} on both sides of the equation $Av = b$, you should be able to derive an expression to solve for v. Write down this expression.*

Premultiplying by A^{-1} :

$$A^{-1}Av = A^{-1}b$$

Since $A^{-1}A = I$, we have $v^* = A^{-1}b$.

(d) (2 pts) *Using the expression in (c) solve for v^*.*

$$v^* = -\frac{1}{5} \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 0 \\ -18+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

5. (14 pts) You are given the following inter-temporal utility function:

$$U = U(c_1, c_2) = \ln c_1 + \beta \ln c_2 \tag{1}$$

where c_1 and c_2 is consumption in period 1 and 2, respectively. $0 < \beta < 1$ is the rate at which you discount the future and it measures your impatient. You earn income $y_1 > 0$ in period 1 and income $y_2 > 0$ in period 2. Any of the income you save s in period 1 earns interest $r > 0$. So,

$$c_1 + s = y_1, \qquad c_2 = y_2 + (1 + r)s$$

Combining these constraints:

$$c_1 + \frac{1}{1+r} c_2 = y_1 + \frac{1}{1+r} y_2$$

Let the present-discounted income be denoted by m, such that:

$$m = y_1 + \frac{1}{1+r} y_2$$

You want to choose c_1 and c_2 to maximize utility $U(c_1, c_2)$ in equation (1) subject to the constraint:

$$c_1 + \frac{1}{1+r} c_2 = m \tag{2}$$

(a) (2 pts) Write down the Lagrangian function corresponding to this maximization problem.

$$L(c_1, c_2, \lambda) = \ln c_1 + \beta \ln c_2 + \lambda \left(m - c_1 - \frac{1}{1+r} c_2 \right)$$

(b) (3 pts) Write down the first-order conditions for a critical point.

$$\frac{\partial L}{\partial c_1} = \frac{1}{c_1^*} - \lambda^* = 0 \tag{3}$$

$$\frac{\partial L}{\partial c_2} = \frac{\beta}{c_2^*} - \frac{\lambda^*}{1+r} = 0 \tag{4}$$

$$\frac{\partial L}{\partial \lambda} = m - c_1^* - \frac{1}{1+r}c_2^* = 0 \tag{5}$$

(c) (2 pts) Using the first order conditions in (b), show that the optimal consumption c_1^* and c_2^* and the Lagrange multiplier λ^* are given by:

$$c_1^* = \frac{m}{1+\beta}, \quad c_2^* = \frac{\beta m(1+r)}{1+\beta}, \quad \lambda^* = \frac{1+\beta}{m}$$

Note that from (3), $\lambda^* = 1/c_1^*$, plugging this in (4), we get:

$$c_2^* = \beta(1+r)c_1^*$$

Plugging this expression for c_2^* in (5):

$$c_1^* + \frac{1}{1+r}\beta(1+r)c_1^* = m \rightarrow c_1^* = \frac{m}{1+\beta}$$

In which case,

$$c_2^* = \beta(1+r)c_1^* = \frac{\beta m(1+r)}{1+\beta}$$

Finally, since $\lambda^* = 1/c_1^*$,

$$\lambda^* = \frac{1}{c_1^*} = \frac{1+\beta}{m}$$

(d) (1 pt) Here, $U(c_1, c_2)$ is a strictly quasiconcave function. Is this sufficient to conclude that the c_1^* and c_2^* we found in (c) characterize a global maximum? Answer: Yes

(e) (2 pts) How does the optimal consumption in period 1 and 2 change due to an increase in m? Calculate $\partial c_1^*/\partial m$ and $\partial c_2^*/\partial m$ to answer your question.

$$\frac{\partial c_1^*}{\partial m} = \frac{1}{1+\beta} > 0, \qquad \frac{\partial c_2^*}{\partial m} = \frac{\beta(1+r)}{1+\beta} > 0$$

Since $\beta > 0, r > 0$, both $\partial c_1^*/\partial m$ and $\partial c_2^*/\partial m$ are positive and hence optimal consumption increases in both periods due to an increase in m. In particular, c_1^* increases by $1/(1+\beta)$ for a one unit increase in m. While, c_2^* increases by $\beta(1+r)/(1+\beta)$ for a one unit increase in m.

(f) (1 pt) If $r = 0$, using your expressions for $\partial c_1^*/\partial m$ and $\partial c_2^*/\partial m$, answer whether optimal consumption in period 1 increases by more or less than consumption in period 2 in response to a change in m? When $r = 0$,

$$\frac{\partial c_1^*}{\partial m} = \frac{1}{1+\beta} > \frac{\beta}{1+\beta} = \frac{\partial c_2^*}{\partial m}$$

Since $\beta < 1$ (consumption in the future is discounted), when the interest rate $r = 0$, optimal consumption in period 2 increases by less relative to period 1 due to a one unit increase in m.

(g) (2 pts) How does optimal consumption in period 1 change due to an increase in the interest rate r?

Note that here,

$$m = y_1 + \frac{y_2}{1+r}$$

So to calculate $\partial c_1^*/\partial r$, you need to use the chain-rule as follows:

$$\frac{\partial c_1^*}{\partial r} = \frac{\partial c_1^*}{\partial m} \cdot \frac{\partial m}{\partial r} = \frac{1}{1+\beta} \cdot \frac{-y_2}{(1+r)^2} < 0$$

So optimal consumption in period 1 decreases due to an increase in interest rate r.